New insights on the intuitionistic interpretation of Default Logic

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Abstract. An interesting result found by Truszczyński showed that (non-monotonic) modal logic S4F can be used to encode Default Logic (DL). In this work we further investigate the relation between both formalisms using Gödel's pattern for translation of Intuitionistic Logic into S4. This pattern not only allows encoding DL into S4F but also preserves this feature for two general nonmonotonic formalisms: Turner's *Nested Default Logic* (NDL) and Pearce's *Equilibrium Logic* (which encodes logic programs into the intermediate logic of Here-and-There). For comparison purposes, we define a variation of DL (inside S4F) we have called *Intuitionistic Default Logic* that generalizes both NDL and Equilibrium Logic, in the sense that the former does not allow nesting or combining the rule conditional operator, whereas the latter exclusively restricts the shape of classical formulas to atoms. Finally, we also prove that the S4F-equivalence of the modal encodings is a necessary and sufficient condition for strong equivalence of IDL default theories.

1 INTRODUCTION

The search for logical encodings of formalisms for Nonmonotonic Reasoning (NMR) has played a relevant role in that research area. While some nonmonotonic formalisms are defined in logical terms from the very beginning (think, for instance, in nonmonotonic modal logic [13]), there exists a considerable effort in logically capturing other popular frameworks like Reiter's Default Logic (DL) [17] or Logic Programming (LP) under *stable models* semantics [3]. The main advantage of a logical encoding is that it provides a clear, fully semantic interpretation for all the constructs handled in the NMR formalism, pointing out new directions for possible generalizations. As an important result, the study of properties of the NMR formalism can be translated into theorem proving inside the underlying logical framework.

In the case of DL, a whole family of nonmonotonic modal characterizations (see [12]) is applicable. As for LP, the most relevant encoding is, perhaps, Pearce's *Equilibrium Logic* [15], which consists in a particular models minimization for Heyting's intermediate logic of Here-and-There (HT). This HT characterization has also introduced another interesting topic in NMR: the property of *strong equivalence*. We say that two theories (or logic programs) T_1 and T_2 are *strongly equivalent* when $T_1 \cup T$ and $T_2 \cup T$ yield the same consequences, for any additional set of formulas T. Note that, when dealing with a nonmonotonic consequence relation, this condition is stronger than simple equivalence between T_1 and T_2, since the addition of T may retract consequences in a different way for each theory. It is important to observe that, once we obtain an encoding for DL or LP relying on some logic $\mathcal{L}$, equivalence of T_1 and T_2 translations into $\mathcal{L}$ usually provides a sufficient condition for strong equivalence, but perhaps not a necessary one. Our claim is that this last feature is a *desirable property* for any logical encoding.

In an outstanding paper [9], Lifschitz, Pearce and Valverde proved that HT passes this test, that is, HT-equivalence is a necessary and sufficient condition for strong equivalence of logic programs. This paper was followed by several adaptations that use instead classical propositional logic [16, 11], three-valued logic [1] or, in a more general result [2], any intermediate logic satisfying the axiom of weak excluded middle. In all these cases, the result is still applicable for the general syntax of *nested* LP [10], where default negation, conjunction and disjunction can be freely combined both in the head and the body of rules. Strong equivalence for default theories has been studied by Turner in [19], using a non-logical approach inspired by the LP case in [9]. In fact, Turner presents his result for an analogous generalization of DL called *Nested Default Logic* (essentially, nested LP where atoms can be replaced by classical formulas).

In this paper, we study some interesting properties of the translation of DL into (non-monotonic) modal logic S4F. This translation was introduced by Truszczyński in [18] and was later observed to follow Gödel's general pattern [5] for encoding[2] Intuitionistic Logic into S4. In our work, the straightforward application of Gödel's pattern allows us to propose one more generalization of DL we have called *Intuitionistic Default Logic* (IDL), and which we show extremely useful for comparison purposes. This generalization consists in freely combining intuitionistic operators, but allowing classical formulas to play the role of "atoms." As a result, we can prove that: (1) IDL generalizes NDL, in the sense that it does not impose any restriction for combining intuitionistic operators (curiously, despite of its name, NDL does not allow nesting default rules); (2) equivalence of IDL theories (under S4F) is a necessary and sufficient condition for strong equivalence of default theories; and (3), IDL is a proper generalization of HT, when we consider classical formulas instead of atoms.

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[2] Gödel's pattern has been frequently used for discovering modal companions of intermediate logics. Its application to nonmonotonic formalisms is first explicitly mentioned in [14], which captures LP under answer sets semantics into nonmonotonic S4.

The rest of the paper is organized as follows. Section 2 contains some definitions and notation, together with a brief recall of modal logic. The next section introduces a nonmonotonic version of S4F and proceeds then to define IDL inside this framework. Sections 4 and 5 study the relation to NDL and HT, respectively. Finally, Section 6 concludes the paper and informally outlines other connections to related work. An unpublished, extended version of this document[3] contains the proofs for the main results.

2 PRELIMINARIES

All languages described in the paper are assumed to be propositional (variables are used as abbreviations of all their ground instances). We will continuously refer to a set of finite ground atoms At which will be frequently used as propositional signature. We adopt the following notation for languages:

$$[op_1\ op_2\ \dots\ op_n](At)$$

standing for all the formulas constructed with elements of signature At combined with operators op_i. Note that signature can be different from At: some languages can use as "atoms" the set of formulas of another language. Operators $[\vee\ \wedge\ \ \supset\ \equiv\ \ \neg\ \bot\ \top]$ will be called *classical* and defined with their standard arity and precedence. We will also use a second set of operators $[not\ \ ,\ ;\ \leftarrow]$ called *intuitionistic* and inherited from LP syntax with the usual arity and precedence (',' stands for conjunction and ';' for disjunction). The reason for the name "intuitionistic" is that, in fact, we will use the same notation for logic programs, default theories and intuitionistic logic, in order to reduce translation efforts. When combined, we assume that classical operators have higher priority than intuitionistic ones.

The set of *classical formulas*, $\mathcal{L}$, is defined as $[\vee\ \neg\ \bot](At)$. (the rest of classical operators are derived from the previous ones in the usual way). We will use capital letters $F, G, \dots$ to denote classical formulas.

Given a set of atoms At, a *propositional interpretation*, I, is represented as any subset $I \subseteq At$ pointing out all the atoms valuated as *true*. If S is a set of propositional interpretations, then we define $Th(S)$ as the set of classical formulas that are satisfied by all interpretations in S:

$$Th(S) \stackrel{\text{def}}{=} \{F \in \mathcal{L} \mid \text{for all } I \in S, I \models F\}$$

For the following definitions regarding modal logic, we have mostly followed [12]. The language of *modal formulas* is defined as $\mathcal{L}_L = [\vee\ \neg\ \bot\ L](At)$, that is, classical operators plus an additional unary operator L, called the *necessity* functor. The dual operator M, called *possibility*, is derived from L as $M\phi \stackrel{\text{def}}{=} \neg L \neg \phi$. A *modal theory* will be any subset of $\mathcal{L}_L$.

A *modal logic* $\mathcal{S}$ is usually described in terms of a set of axioms. We write $T \vdash_{\mathcal{S}} \phi$ to express that formula ϕ is *derivable* from T and axioms of $\mathcal{S}$ using the inference rules of *modus ponens* (MP) and *necessitation* (N):

$$\frac{\phi, \phi \supset \psi}{\psi}\ \ (\text{MP}) \qquad\qquad \frac{\vdash_{\mathcal{S}} \phi}{\vdash_{\mathcal{S}} L\phi}\ \ (\text{N})$$

We define the *consequences* of any modal theory T under logic $\mathcal{S}$ as $Cn_{\mathcal{S}}(T) \stackrel{\text{def}}{=} \{\phi \in \mathcal{L}_L \mid T \vdash_{\mathcal{S}} \phi\}$.

[3] Reference to online draft has been omitted for blind reviewing.

We will be particularly interested in the following set of axioms:

- **k**. $L(\phi \supset \psi) \supset (L\phi \supset L\psi)$
- **t**. $L\phi \supset \phi$
- **4**. $L\phi \supset LL\phi$
- **f**. $\phi \wedge ML\psi \supset L(M\phi \vee \psi)$
- **5**. $M\phi \supset LM\phi$

Modal logic S4 corresponds to the set of axioms $\{\mathbf{k}, \mathbf{t}, \mathbf{4}\}$, logic S4F is defined as S4+$\{\mathbf{f}\}$ whereas S5 corresponds to S4+$\{\mathbf{5}\}$.

The semantics of these three logics can be captured in terms of the so-called Kripke models. A *Kripke model* is a triple $\mathcal{M} = \langle W, R, V \rangle$ where W is a nonempty set (whose elements are the *worlds* of $\mathcal{M}$), $R \subseteq W \times W$ is called the *accessibility relation* among worlds, and finally, V is a set of propositional interpretations, one I_w for each world $w \in W$. We define when a Kripke model $\mathcal{M}$ *satisfies* a modal formula ϕ at a given world w, written $(\mathcal{M}, w) \models \phi$, recursively as follows:

1. $(\mathcal{M}, w) \models p$ iff $p \in I_w$ for any atom p.
2. $(\mathcal{M}, w) \models \neg\phi$ iff $(\mathcal{M}, w) \not\models \phi$.
3. $(\mathcal{M}, w) \models \phi \vee \psi$ iff $(\mathcal{M}, w) \models \phi$ or $(\mathcal{M}, w) \models \psi$.
4. $(\mathcal{M}, w) \models L\phi$ iff for all $w' \in W$ s.t. wRw', $(\mathcal{M}, w') \models \phi$.

When ϕ is satisfied at any world w of $\mathcal{M}$ we simply write $\mathcal{M} \models \phi$ and usually say that ϕ is *valid* in $\mathcal{M}$. It is not difficult to see that:

Proposition 1 *If relation R is reflexive:* $\mathcal{M} \models \phi$ *iff* $\mathcal{M} \models L\phi$. □

Similarly, a modal theory T is *valid* in $\mathcal{M}$, also written $\mathcal{M} \models T$, when all the formulas in T are valid in $\mathcal{M}$. Given a class of Kripke models $\mathcal{K}$, we can define *entailment*, $\models_{\mathcal{K}}$, as follows. For any theory T and formula ϕ, we write $T \models_{\mathcal{K}} \phi$ to express that any $\mathcal{M} \in \mathcal{K}$ such that $\mathcal{M} \models T$ satisfies $\mathcal{M} \models \phi$. As expected, $\models_{\mathcal{K}} \phi$, means that ϕ is true in any Kripke model of class $\mathcal{K}$.

A modal logic $\mathcal{S}$ is said to be *characterized* by a class of Kripke models $\mathcal{K}$ iff deduction and entailment coincide, that is, for any T and ϕ: $T \vdash_{\mathcal{S}} \phi$ iff $T \models_{\mathcal{K}} \phi$.

The class of Kripke models characterizing S4 consists of those with a transitive and reflexive accessibility relation. Kripke models for S5 have the shape $\langle W, W \times W, V \rangle$ (that is, they are transitive, reflexive and symmetric). Usually, S5-models are directly represented as $\langle W, V \rangle$. Finally, the most interesting structure for our purpose is the class of Kripke models characterizing S4F, which have the shape $\langle W, (W_1 \times W) \cup (W \times W_2), V \rangle$ where $W = W_1 \cup W_2$, $W_1 \cap W_2 = \emptyset$ and $W_2 \neq \emptyset$. In other words, each S4F-model consists of a pair of S5 clusters, W_1 and W_2, where W_1 is fully connected to W_2. We will directly represent the S4F-model as $\langle W_1, W_2, V \rangle$. Note that, when $W_1 = \emptyset$, we can consider that it actually amounts to an S5-model $\langle W_2, V \rangle$.

It is perhaps interesting to note that the number of different modalities in each of the three mentioned logics is relatively small. By *different modality* we mean a string of modal operators which cannot be equivalently reduced into a smaller string. It is well-known [7] that S4 has the following six different modalities[4] L, M, LM, ML, LML, MLM. For instance, in

[4] We omit everywhere the case of non-modal formulas, which could also be considered as an additional "empty" modality.

S4 we have:

$$LL\phi \equiv L\phi \quad (1)$$
$$MM\phi \equiv M\phi \quad (2)$$

In S5, this number of modalities is reduced to just two: L and M. In a similar way, the following theorem:

$$ML\phi \supset LM\phi \quad (3)$$

from S4F can be used to prove:

$$LML\phi \equiv ML\phi \quad (4)$$
$$MLM\phi \equiv LM\phi \quad (5)$$

showing that this logic has the four modalities L, M, LM and ML. In fact, we can just consider L and ML, seeing M and LM as their respective negations. The following theorems of S4F describe some unfolding properties of these modalities which will be especially useful later:

$$
\begin{aligned}
L\neg ML\phi &\equiv \neg ML\phi && (6)\\
L(\phi \wedge \psi) &\equiv L\phi \wedge L\psi && (7)\\
L(L\phi \vee L\psi) &\equiv L\phi \vee L\psi && (8)\\
L(L\phi \supset L\psi) &\equiv (L\phi \supset L\psi) \wedge (ML\phi \supset ML\psi) && (9)\\
ML\neg ML\phi &\equiv \neg ML\phi && (10)\\
ML(\phi \wedge \psi) &\equiv ML\phi \wedge ML\psi && (11)\\
ML(L\phi \vee L\psi) &\equiv ML\phi \vee ML\psi && (12)\\
ML(L\phi \supset L\psi) &\equiv ML\phi \supset ML\psi && (13)
\end{aligned}
$$

3 NONMONOTONIC S4F

The most usual way of defining a nonmonotonic version of a modal logic is using McDermott and Doyle's fixpoint definition [13] of the concept of *expansion*. Given a modal logic $\mathcal{S}$, we say that theory E is an *$\mathcal{S}$-expansion* of theory T iff E is consistent with $\mathcal{S}$ and satisfies: $E = Cn_{\mathcal{S}}(T \cup \{\neg L\phi \mid \phi \notin E\})$.

In this work, however, we propose a different characterization in terms of minimal models. To this aim, we begin defining for any S4F model, an associated pair of sets formulas. We call *candidate set* to any consistent, logically closed set of classical formulas. For any S4F-model $\mathcal{M} = \langle W_1, W_2, V \rangle$ we define the pair of candidate sets $(H_{\mathcal{M}}, T_{\mathcal{M}})$ as:

$$H_{\mathcal{M}} \stackrel{\text{def}}{=} Th(I_w \mid w \in W_1 \cup W_2)$$
$$T_{\mathcal{M}} \stackrel{\text{def}}{=} Th(I_w \mid w \in W_2)$$

Looking at their definition, it is clear that $H_{\mathcal{M}} \subseteq T_{\mathcal{M}}$. A possible interpretation of $(H_{\mathcal{M}}, T_{\mathcal{M}})$ is that it describes the agent's beliefs in a partial way: she *believes* all formulas in $H_{\mathcal{M}}$ whereas she *does not believe* any formula not in $T_{\mathcal{M}}$. Thus, uncertainty comes from the fact that the agent has no particular belief with respect to formulas in $T_{\mathcal{M}} - H_{\mathcal{M}}$.

This structure has a straightforward correspondence with modalities in S4F, as asserted by the following theorem:

Theorem 1 *For any classical formula F:*
$F \in H_{\mathcal{M}}$ iff $\mathcal{M} \models LF$ and
$F \in T_{\mathcal{M}}$ iff $\mathcal{M} \models MLF$. □

Definition 1 (Ordering relation $\leq$) Given two S4F models $\mathcal{M}$ and $\mathcal{M}'$, we say that $\mathcal{M} \leq \mathcal{M}'$ iff $T_{\mathcal{M}} = T_{\mathcal{M}'}$ and $H_{\mathcal{M}} \subseteq H_{\mathcal{M}'}$. □

In other words, $\leq$-minimal models correspond to fixing the non-believed formulas and minimizing the believed ones. The final selected models correspond to minimal models in which the set of beliefs contain no uncertainty:

Definition 2 (Selected model) A S4F-model of a theory T is said to be *selected* iff $H_{\mathcal{M}} = T_{\mathcal{M}}$ and there is no other $\mathcal{M}' \models T$ such that $M' < M$. □

3.1 Intuitionistic Default Logic

We can now define an interesting subset of nonmonotonic S4F by restricting the use of modal operators in the following way. The language of *Intuitionistic Default Logic* (IDL) corresponds to $[not\ ,\ ;\ \leftarrow](\mathcal{L})$. In other words, we construct formulas with intuitionistic operators but using the set of classical formulas as "atoms". The name IDL should not lead to confusion: it is not an intuitionistic *variant* of Default Logic, but an intuitionistic *interpretation* of default constructs instead. We say that IDL is a subset of S4F because intuitionistic connectives are actually defined in terms of modal expressions, directly following Gödel's translation [5]:

$$
\begin{aligned}
(not\ \phi) &\stackrel{\text{def}}{=} L\neg L\phi\\
(\phi, \psi) &\stackrel{\text{def}}{=} \phi \wedge \psi\\
(\phi; \psi) &\stackrel{\text{def}}{=} L\phi \vee L\psi\\
(\phi \leftarrow \psi) &\stackrel{\text{def}}{=} L\psi \supset L\phi
\end{aligned}
$$

The translation of negation[5] is equivalent to $\neg ML\phi$. Thus, by Theorem 1, an intuitive interpretation of $(not\ F)$ is $F \notin T_{\mathcal{M}}$, that is, F is *not believed* by the agent. Using this translation, it is not difficult to show that the following equivalences are theorems in S4F:

$$
\begin{aligned}
not\ not\ not\ \phi &\equiv not\ \phi && (14)\\
not\ (\phi, \psi) &\equiv (not\ \phi; not\ \psi) && (15)\\
not\ (\phi; \psi) &\equiv (not\ \phi, not\ \psi) && (16)
\end{aligned}
$$

The importance of these properties is that they show that formulas of sub-language $[not\ ,\ ;](\mathcal{L})$ can be unfolded until occurrences of *not* have the shape $(not\ F)$ or $(not\ not\ F)$, being F a classical formula.

It should perhaps be observed that, due to Proposition 1, requiring $\mathcal{M} \models \phi$ in IDL is the same than $\mathcal{M} \models L\phi$, and so, all expressions are implicitly in the scope of a necessity operator. The way in which this L operator can be unfolded with respect to intuitionistic operators is described by equivalences (1), (2) and (4)-(13).

Using LP notation for intuitionistic operators helps establish a direct syntactic correspondence with most classes of logic programs. For this reason, we have preferred to maintain the use of LP conjunction (ϕ, ψ), although as seen above,

[5] For translating *not* ϕ, Gödel actually proposed a second alternative $\neg L\phi$. Although most results in the paper are still valid under this choice, we have preferred the stronger version $L\neg L\phi$ in order to obtain simpler proofs and provide a more direct interpretation of negation in terms of $T_{\mathcal{M}}$.

it does not differ from classical conjunction. In the case of variants of default theories the correspondence for our notation is not so straightforward, although it can be easily deduced. A (disjunctive) default rule like:

$$\frac{A : B_1, \ldots, B_n}{C_1 | \ldots | C_m}$$

would be represented in IDL as:

$$C_1; \ldots ; C_m \leftarrow A, \mathit{not}\ \neg B_1, \ldots, \mathit{not}\ \neg B_n$$

The following property can be easily checked:

Property 1 *For any IDL formula ϕ, if $\langle W_1, W_2, V \rangle \models \phi$ then $\langle W_2, V \rangle \models \phi$.* □

That is, if a S4F model satisfies an IDL formula ϕ, then the S5-model just consisting of cluster W_2 also satisfies ϕ.

4 NESTED DEFAULT LOGIC

The syntax of Nested Default Logic (NDL) is a subset of IDL where connective '$\leftarrow$' cannot be in the scope of other operator. Therefore, a NDL theory is a set of *rules* like $\phi \leftarrow \psi$, where ϕ and ψ belong to language $[\mathit{not}\ ,\ ;](\mathcal{L})$ (called the set of *NDL formulas*). Classical formulas can be included in the NDL theory as rules like $F \leftarrow \top$.

The *satisfaction* of an NDL formula F by a candidate set X is denoted as $X \models_{\mathrm{ND}} \phi$ and recursively defined as follows:

- $X \models_{\mathrm{ND}} F$ iff $F \in X$, for any classical formula F.
- $X \models_{\mathrm{ND}} (\phi, \psi)$ iff $X \models_{\mathrm{ND}} \phi$ and $X \models_{\mathrm{ND}} \psi$
- $X \models_{\mathrm{ND}} (\phi; \psi)$ iff $X \models_{\mathrm{ND}} \phi$ or $X \models_{\mathrm{ND}} \psi$
- $X \models_{\mathrm{ND}} \mathit{not}\ \phi$ iff $X \not\models_{\mathrm{ND}} \phi$

As expected, X is a *model* of a NDL theory D, also written $X \models_{\mathrm{ND}} D$, when $X \models_{\mathrm{ND}} \psi$ implies $X \models_{\mathrm{ND}} \phi$, for any rule $\phi \leftarrow \psi$ in D.

The *reduct* of a NDL formula ϕ with respect to X, denoted as ϕ^X, is the result of replacing any maximal[6] subformula $\mathit{not}\ \psi$ either by $\bot$ or $\top$ depending on whether $X \models_{\mathrm{ND}} \phi$ or not, respectively. The *reduct* of a default theory D, written D^X, is obtained by replacing each rule $\phi \leftarrow \psi$ in D by $\phi^X \leftarrow \psi^X$.

Definition 3 (Extension) A candidate set X is an *extension* of a default theory D iff X is a minimal (w.r.t. set inclusion) model of D^X. □

In [19] it is shown that NDL properly generalizes Reiter's Default Logic and its extension for disjunctive defaults introduced in [4]. The following theorem shows that IDL semantics covers, in its turn, NDL extensions:

Theorem 2 *Let D be a NDL theory, X a candidate set and $\mathcal{M}$ a S4F model for which $H_{\mathcal{M}} = T_{\mathcal{M}} = X$. Then, X is an extension of D iff $\mathcal{M}$ is a selected model for D under IDL.* □

The structure of a single candidate set does not suffice, however, for capturing the property of strong equivalence of default theories.

[6] That is, any subformula ($\mathit{not}\ \psi$) of ϕ which is not, in its turn, in the scope of an outer not .

Definition 4 (SE-model) We define a *SE-model* of some default theory D as a pair (X, Y) of candidate sets satisfying $X \subseteq Y$, $X \models_{\mathrm{ND}} D^Y$ and $Y \models_{\mathrm{ND}} D^Y$. □

The idea of handling these two sets is that we will take into account both the "initial" candidate set Y used for getting the reduct, and the "resulting" candidate sets X that are models of the reduct.

Proposition 2 (From Theorem 3 in [19]) *Two NDL theories are strongly equivalent iff they have the same SE-models.* □

Now, the following theorem is essential for adapting this result for IDL:

Theorem 3 *Let (X, Y) be a pair of candidate sets with $X \subseteq Y$ and let $\mathcal{M}$ be some S4F model such that $X = H_{\mathcal{M}}$ and $Y = T_{\mathcal{M}}$. Then, for any NDL theory D, (X, Y) is an SE-model of D iff $\mathcal{M} \models D$ in IDL.* □

This directly means that we can rephrase now Proposition 2 so that strong equivalence of NDL theories corresponds to S4F-equivalence of their modal translations. In fact, this result is even more general. Since Turner's proof for Proposition 2 exclusively deals with sets of SE-models (without reference to the syntax of their original theories), and thanks to correspondence established in Theorem 2, it is not difficult to see that:

Corollary 1 *Two IDL theories are strongly equivalent iff their modal encodings are S4F equivalent.* □

5 HERE-AND-THERE

The previous section has shown that IDL generalizes NDL which, in its turn, is a generalization of nested LP. On the other hand, as said in the introduction, Lifschitz, Pearce and Valverde [9] showed that the HT encoding of LP also captures nested expressions. However, the HT encoding is still applicable to more general expressions than nested LP syntax, providing an intuitive meaning to constructions in which the rule arrow is inside the scope of other operator. For instance, in HT we have:

$$(p \leftarrow q) \leftarrow r \quad \Leftrightarrow \quad p \leftarrow q, r$$
$$\mathit{not}\ (p \leftarrow q) \quad \Leftrightarrow \quad (\bot \leftarrow \mathit{not}\ q),\ (\bot \leftarrow p)$$

where $\Leftrightarrow$ stands for semantic equivalence. This nice feature may be lost for other encodings, like shown for instance in [1], for the case of three-valued logic.

The question now is, when we move to consider classical formulas instead of atoms, does IDL provide an intuitive meaning for this type of constructions? In this section we show that, in fact, the monotonic basis of IDL (that is, its S4F translation) is a proper generalization of HT. In other words, Gödel's pattern for translating intuitionistic logic into S4 is also valid for translating HT into S4F.

We use the language $[\mathit{not}\ ,\ ;\ \leftarrow\ \bot](At)$, called *intuitionistic formulas*, for describing the syntax of Here-and-There (HT). We will understand $\mathit{not}\ \phi$ as an abbreviation of $\bot \leftarrow \phi$. The semantics of HT is described as follows. An HT *world* is any element of the set $\{h, t\}$ (respectively standing for *here* and *there*). We define an accessibility relation $\preceq$ so that $h \preceq h$, $t \preceq t$ and $h \preceq t$.

Definition 5 (HT-interpretation) Given a propositional signature At, an HT *interpretation* is defined as the pair $\mathcal{I} = (I^h, I^t)$ where $I^h \subseteq I^t \subseteq At$. □

The HT interpretation can be understood as a partial truth valuation for atoms in the signature. Intuitively, I^h contains the true atoms, $\Sigma - I^t$ the false atoms and, finally, $I^t - I^h$ corresponds to those atoms that are left undefined. An interpretation of shape (I, I) is said to be *total* (there are no undefined atoms).

Definition 6 (Satisfaction of a formula) We recursively define the *satisfaction* of a formula ϕ by an interpretation $\mathcal{I} = (I^h, I^t)$ at a world w, written $(\mathcal{I}, w) \models_{\text{HT}} \phi$, in the following way:

1. $(\mathcal{I}, w) \models_{\text{HT}} p$ iff $p \in I^w$
2. $(\mathcal{I}, w) \models_{\text{HT}} (\phi, \psi)$ iff $(\mathcal{I}, w) \models_{\text{HT}} \phi$ and $(\mathcal{I}, w) \models_{\text{HT}} \psi$
3. $(\mathcal{I}, w) \models_{\text{HT}} (\phi; \psi)$ iff $(\mathcal{I}, w) \models_{\text{HT}} \phi$ or $(\mathcal{I}, w) \models_{\text{HT}} \psi$
4. $(\mathcal{I}, w) \models_{\text{HT}} (\psi \leftarrow \phi)$ iff for all w' such that $w \preceq w'$, $(\mathcal{I}, w') \not\models_{\text{HT}} \phi$ or $(\mathcal{I}, w) \models_{\text{HT}} \psi$
5. $(\mathcal{I}, w) \not\models_{\text{HT}} \bot$

□

We say that an HT interpretation $\mathcal{I}$ is a *model* of a theory T iff $(\mathcal{I}, h) \models_{\text{HT}} \phi$ for all ϕ in T. The following property of HT corresponds, somehow, to Property 1 for IDL:

Property 2 *For all intuitionistic formula ϕ, if $(\mathcal{I}, h) \models \phi$ then $(\mathcal{I}, t) \models \phi$.* □

Definition 7 *Given a S4F model $\mathcal{M}$ we define the corresponding HT interpretation $\mathcal{I}_{\mathcal{M}} = (I^h_{\mathcal{M}}, I^t_{\mathcal{M}})$ as:*

$$I^h_{\mathcal{M}} \stackrel{\text{def}}{=} \bigcap_{w \in W_1 \cup W_2} I_w \qquad\qquad I^t_{\mathcal{M}} \stackrel{\text{def}}{=} \bigcap_{w \in W_2} I_w$$

□

In other words, $I^h_{\mathcal{M}}$ (resp. $I^t_{\mathcal{M}}$) collects all the atoms included in $H_{\mathcal{M}}$ (resp. $T_{\mathcal{M}}$). Note that different S4F models may lead to the same $I_{\mathcal{M}}$.

Lemma 1 *Let ϕ be an intuitionistic formula, $\mathcal{M} = \langle W_1, W_2, V\rangle$ an S4F model and $I_{\mathcal{M}}$ its corresponding HT interpretation. Then:*
(a) $(\mathcal{I}_{\mathcal{M}}, h) \models_{HT} \phi$ *iff* $\mathcal{M} \models \phi$.
(b) $(\mathcal{I}_{\mathcal{M}}, t) \models_{HT} \phi$ *iff* $\langle W_2, V\rangle \models \phi$. □

Theorem 4 *For any intuitionistic formula ϕ:*
$\models_{HT} \phi$ *iff* $\models \phi$ *under S4F.* □

6 CONCLUSION

We have deepen into the S4F encoding of Default Logic, showing some interesting properties that point out its general utility. Following Gödel's translation of intuitionistic logic into S4, we presented a subset of S4F, we have called Intuitionistic Default Logic (IDL), with the aim of showing that other general nonmonotonic formalisms like Turner's Nested Default Logic (NDL) or Pearce's Equilibrium Logic can be seen as particular cases of the S4F characterization. Besides, we adapted NDL results to show that two default theories are strongly equivalent iff their encodings are S4F-equivalent.

As a result, we obtain some important advantages with respect to NDL. As an example, we can show properties about default theories in terms of theorem proving in S4F, what can be automated with a tableaux-style prover (like the one proposed in [6], for instance). Note that, on the contrary, when we directly use NDL definitions, automated reasoning is not straightforward: (meta)proofs for properties are *ad hoc*, using non-logical constructs (like the theory reduct) and logically closed sets of formulas.

We have also seen that IDL still provides a meaning for expressions where the rule conditional is in the scope of other operators. Thus, we can really *nest default rules*, something not possible in NDL, and preserve the same meaning than the one provided by Here-and-There with respect to logic programs. In fact, the relation we established between S4F and Here-and-There, allows using modal S4F provers for proving theorems in that intermediate logic.

Some open topics are left for future work. For instance, it remains to prove that the S4F models minimization proposed in this paper actually corresponds to the standard McDermott and Doyle's syntactic fixpoint definition. Another interesting topic is the relation to the bimodal logic of *Minimal Belief and Negation as Failure* (MBNF) [8].

Logic MBNF deals with two *independent* modalities, L (originally denoted as B) and *not* . It seems clear that these two modalities have a straightforward correspondence to the respective S4F modalities L and $\neg ML$. Following this analogy, the formula $ML\phi$ in S4F would play the same role as *not not* ϕ in MBNF. However, the main difference is that, in MBNF, there is no a priori connection between L and *not* . As a result, S4F is stronger the sense that, for obtaining similar results, we should add to MBNF the axiom $L\phi \supset$ *not not* ϕ and translate each rule operator as:

$$\phi \leftarrow \psi \quad \stackrel{\text{def}}{=} \quad (L\psi \supset L\phi) \wedge (\textit{not not } \psi \supset \textit{not not } \phi)$$

Our conjecture is that this weakness of MBNF will prevent to obtain a necessary condition for strong equivalence of default theories when using this logic.

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APPENDIX A

Proofs of modal formulas

In these proofs we use some results directly extracted from [7], preserving the original reference codes (we write them in bold face) used in the book. This results are classified with respect to the weaker modal system in which they can be proved (this explains the first letters in their code K, T, and S4):

K3. $L(\phi \wedge \psi) \equiv L\phi \wedge L\psi$
K5. $L\phi \equiv \neg M \neg \phi$
K6. $M(\phi \vee \psi) \equiv M\phi \vee M\psi$
K8. $M(\phi \wedge \psi) \supset M\phi \wedge M\psi$
K9. $L(\phi \vee \psi) \supset L\phi \vee M\psi$

T1. $\phi \supset M\phi$

S4(2). $L\phi \equiv LL\phi$
S4(3). $M\phi \equiv MM\phi$

Besides, we use the following inference rules also extracted from [7]:

DR1. $\dfrac{\phi \supset \psi}{L\phi \supset L\psi}$ **DR3**. $\dfrac{\phi \supset \psi}{M\phi \supset M\psi}$

DR4. $\dfrac{\phi \equiv \psi}{M\phi \equiv M\psi}$

In the proofs justifications, the abbreviation "Eq" stands for the rule of substitution of equivalents, "P.C." refers to any theorem from Propositional Calculus, and "M.P." to Modus Ponens. Numbers inside parentheses refer to formulas from this paper, whereas numbers without parentheses refer to proof steps. Finally, $\phi[\alpha/\beta]$ stands for the substitution of β in ϕ by α.

Proof of (1)

It is an S4 theorem referred as **S4(2)** in [7].

Proof of (2)

It is an S4 theorem referred as **S4(3)** in [7].

Proof of (3)

1.	$M\phi \wedge ML\phi \supset L(MM\phi \vee \phi)$	$\mathbf{f}[M\phi/\phi, \phi/\psi]$
2.	$M\phi \wedge ML\phi \supset L(M\phi \vee \phi)$	**S4(3)** × 1 × Eq
3.	$(\phi \supset M\phi) \supset (\phi \vee M\phi \equiv M\phi)$	P.C.
4.	$\phi \vee M\phi \equiv M\phi$	**T1** × 3 × M.P.
5.	$M\phi \wedge ML\phi \supset LM\phi$	4 × 2 × Eq
6.	$ML\phi \supset M\phi$	**t** × **DR3**
7.	$(ML\phi \supset M\phi) \supset (ML\phi \wedge M\phi \equiv ML\phi)$	P.C.
8.	$ML\phi \wedge M\phi \equiv ML\phi$	6 × 7 × M.P.
9.	$ML\phi \supset LM\phi$	5 × 8 × Eq □

Proof of (4)

1.	$LML\phi \supset ML\phi$	$\mathbf{t}[ML\phi/\phi]$
2.	$MLL\phi \supset LML\phi$	$(3)[L\phi/\phi]$
3.	$ML\phi \supset LML\phi$	$\mathbf{S4(2)} \times 2 \times \mathrm{Eq}$
4.	$ML\phi \equiv LML\phi$	$1 \times 3 \times$ P.C. $\square$

Proof of (5)

1.	$ML\neg\phi \equiv LML\neg\phi$	$(4)[\neg\phi/\phi]$
2.	$\neg ML\neg\phi \equiv \neg LML\neg\phi$	$1 \times$ P.C.
3.	$LM\phi \equiv MLM\phi$	$2 \times \mathbf{K5}$ $\square$

Proof of (6)

$$
\begin{array}{lll}
 & L\neg ML\phi & \\
\equiv & \neg MML\phi & \text{(by definition of } M) \\
\equiv & \neg ML\phi & \mathbf{S4(2)} \times Eq \qquad \square
\end{array}
$$

Proof of (7)

It is a theorem of S4, (referred as **K3** in [7]).

Proof of (8)

The left to right direction is just obtained from axiom $\mathbf{t}$ with the replacement $\mathbf{t}[L\phi \lor L\psi/\phi]$. The right to left direction is a theorem of S4:

1.	$L\phi \supset L\phi \lor L\psi$	P.C.
2.	$L\psi \supset L\phi \lor L\psi$	P.C.
3.	$LL\phi \supset L(L\phi \lor L\psi)$	$\mathbf{DR1} \times 1$
4.	$LL\psi \supset L(L\phi \lor L\psi)$	$\mathbf{DR1} \times 2$
5.	$L\phi \supset LL\phi$	$\mathbf{4}$
6.	$L\psi \supset LL\psi$	$\mathbf{4}[\psi/\phi]$
7.	$L\phi \supset L(L\phi \lor L\psi)$	$5 \times 3 \times$ transitivity of $\supset$
8.	$L\psi \supset L(L\phi \lor L\psi)$	$6 \times 4 \times$ transitivity of $\supset$
9.	$L\phi \lor \mathcal{L}\psi \supset L(L\phi \lor L\psi)$	$7 \times 8 \times$ P.C. $\square$

Proof of (9)

The left to right direction is a theorem in system T (that is, the subset of S4 formed with axioms $\mathbf{k}$ and $\mathbf{t}$):

1.	$L(L\phi \supset L\psi) \supset (L\phi \supset L\psi)$	$\mathbf{t}[L\phi \supset L\psi/\phi]$
2.	$L(\neg L\phi \lor L\psi) \supset L\neg L\phi \lor ML\psi$	$\mathbf{K9}[\neg L\phi/\phi, L\psi/\psi]$
3.	$L(L\phi \supset L\psi) \supset \neg ML\phi \lor ML\psi$	P.C. $\times$ $\mathbf{K5}$ $\times$ Eq $\times 2$
4.	$L(L\phi \supset L\psi) \supset (ML\phi \supset ML\psi)$	P.C. $\times$ Eq $\times 3$
5.	$L(L\phi \supset L\psi) \supset$ $(L\phi \supset L\psi) \land (ML\phi \supset ML\psi)$	P.C. $\times 1 \times 4$

For the left to right direction, we first show that:

$$(L\phi \supset L\psi) \land (ML\phi \supset ML\psi) \equiv L\neg L\phi \lor \neg L\phi \land ML\psi \lor L\psi$$

is a theorem of T.

$$
\begin{array}{lll}
 & (L\phi \supset L\psi) \land (ML\phi \supset ML\psi) & \\
\equiv & (\neg L\phi \lor L\psi) \land (\neg ML\phi \lor ML\psi) & \\
\equiv & \neg L\phi \land \neg ML\phi & \\
 & \lor \neg L\phi \land ML\psi & \\
 & \lor L\psi \land \neg ML\phi & \\
 & \lor L\psi \land ML\psi & \\
\equiv & \neg ML\phi & \mathbf{t}[L\phi/\phi] \times \text{P.C.} \\
 & \lor \neg L\phi \land ML\psi & \\
 & \lor L\psi \land \neg ML\phi & \\
 & \lor L\psi & \mathbf{t}[L\psi/\psi] \times \text{P.C.} \\
\equiv & \neg ML\phi \lor \neg L\phi \land ML\psi \lor L\psi & \\
\equiv & L\neg L\phi \lor \neg L\phi \land ML\psi \lor L\psi &
\end{array}
$$

Now, we prove that each one of the disjuncts above implies $L(L\phi \supset L\psi)$.

1.	$\neg L\phi \supset (L\phi \supset L\psi)$	P.C.
2.	$L\neg L\phi \supset L(L\phi \supset L\psi)$	$\mathbf{DR1} \times 1$

1.	$\neg L\phi \land MLL\psi \supset L(M\neg L\phi \lor L\psi)$	$\mathbf{f}[\neg L\phi/\phi, L\psi/\psi]$
2.	$\neg L\phi \land ML\psi \supset L(M\neg L\phi \lor L\psi)$	$\mathbf{S4(2)} \times 1 \times$ Eq
3.	$\neg L\phi \land ML\psi \supset L(\neg LL\phi \lor L\psi)$	$\mathbf{K5} \times 2 \times$ Eq
4.	$\neg L\phi \land ML\psi \supset L(\neg L\phi \lor L\psi)$	$\mathbf{S4(2)} \times 3 \times$ Eq
5.	$\neg L\phi \land ML\psi \supset L(L\phi \supset L\psi)$	P.C. $\times 4$

1.	$L\psi \supset (L\phi \supset L\psi)$	P.C.
2.	$LL\psi \supset L(L\phi \supset L\psi)$	$\mathbf{DR1} \times 1$
3.	$L\psi \supset L(L\phi \supset L\psi)$	$\mathbf{S4(2)} \times 2 \times$ Eq$\square$

Proof of (10)

$$
\begin{array}{lll}
 & ML\neg ML\phi & \\
\equiv & M\neg MML\phi & \\
\equiv & M\neg ML\phi & \text{by } \mathbf{S4(3)} \\
\equiv & \neg LML\phi & \\
\equiv & \neg ML\phi & \text{by (4)} \ \square
\end{array}
$$

Proof of (11)

The left to right direction is a theorem of system K:

1.	$L(\phi \land \psi) \supset L\phi$	P.C. $\times$ $\mathbf{K3}$
2.	$ML(\phi \land \psi) \supset ML\phi$	$\mathbf{DR3} \times 1$
3.	$L(\phi \land \psi) \supset L\psi$	P.C. $\times$ $\mathbf{K3}$
4.	$ML(\phi \land \psi) \supset ML\psi$	$\mathbf{DR3} \times 3$
4.	$ML(\phi \land \psi) \supset ML\phi \land ML\psi$	P.C. $\times 2 \times 4$

For the right to left direction, we will use a vatiarion of theorem **K9** from system K:

$$\mathbf{K9'}. \quad M\phi \land L\psi \supset M(\phi \land \psi)$$

whose proof is included below:

$$
\begin{array}{ll}
 & L(\neg\phi \vee \neg\psi) \supset L\neg\phi \vee M\neg\psi \quad \mathbf{K9}[\neg\phi/\phi, \neg\psi/\psi] \\
\equiv & L\neg(\phi \wedge \psi) \supset \neg M\phi \vee \neg L\psi \\
\equiv & \neg M(\phi \wedge \psi) \supset \neg(M\phi \wedge L\psi) \\
\equiv & (M\phi \wedge L\psi) \supset M(\phi \wedge \psi)
\end{array}
$$

Now, we separate the proof of the right to left direction in two steps, one under the assumption of $L\phi$ (step 3 of the proof below) and the other under the hypothesis of $\neg L\phi$ (step 12).

1.	$ML\psi \wedge L\phi \supset M(L\psi \wedge L\phi)$	**K9'**$[L\psi/\phi, \phi/\psi]$
2.	$ML\psi \wedge L\phi \wedge ML\phi \supset M(L\psi \wedge L\phi)$	P.C. $\times$ 1
3.	$L\phi \wedge ML\psi \wedge ML\phi \supset ML(\psi \wedge L\phi)$	**K3** $\times$ 2
4.	$\neg L\phi \wedge MLL\psi \supset L(M\neg L\phi \vee L\psi)$	**f**$[\neg L\phi/\phi, L\psi/\psi]$
5.	$\neg L\phi \wedge ML\psi \supset L(M\neg L\phi \vee L\psi)$	**S4(2)** $\times$ 4 $\times$ Eq
6.	$\neg L\phi \wedge ML\psi \supset L(\neg LL\phi \vee L\psi)$	**K5** $\times$ 5 $\times$ Eq
7.	$\neg L\phi \wedge ML\psi \supset L(\neg L\phi \vee L\psi)$	**S4(2)** $\times$ 6 $\times$ Eq
8.	$\neg L\phi \wedge ML\psi \wedge ML\phi \supset L(L\phi \supset L\psi) \wedge ML\phi$	P.C. $\times$ 7
9.	$ML\phi \wedge L(L\phi \supset L\psi) \supset M(L\phi \wedge (L\phi \supset L\psi))$	**K9'**$[L\phi/\phi, L\phi \supset L\psi, \psi]$
10.	$ML\phi \wedge L(L\phi \supset L\psi) \supset M(L\phi \wedge L\psi)$	P.C. $\times$ 9
11.	$\neg L\phi \wedge ML\psi \wedge ML\phi \supset M(L\phi \wedge L\psi)$	P.C. $\times$ 8 $\times$ 10
12.	$\neg L\phi \wedge ML\psi \wedge ML\phi \supset ML(\phi \wedge \psi)$	**K3** $\times$ 11
13.	$ML\psi \wedge ML\phi \supset M(L\phi \wedge L\psi)$	P.C. $\times$ 3 $\times$ 12 □

Proof of (12)

It is a theorem of S4:

1.	$L\phi \vee L\psi \equiv L(L\phi \vee L\psi)$	(8)
2.	$M(L\phi \vee L\psi) \equiv ML(L\phi \vee L\psi)$	**DR4** $\times$ 1
3.	$M(L\phi \vee L\psi) \equiv ML\phi \vee ML\psi$	**K6**$[L\phi/\phi, L\psi/\psi]$
4.	$ML(L\phi \vee L\psi) \equiv ML\phi \vee ML\psi$	P.C. $\times$ 2 $\times$ 3 □

Proof of (13)

The right to left direction is a theorem of S4 that can be proved as follows:

1.	$M(L\phi \wedge M\neg\psi) \supset ML\phi \wedge MM\neg\psi$	**K8**$[L\phi/\phi, M\neg\psi/\psi]$
2.	$M(L\phi \wedge M\neg\psi) \supset ML\phi \wedge M\neg\psi$	**S4(3)** $\times$ 1 $\times$ Eq
3.	$LM(L\phi \wedge M\neg\psi) \supset L(ML\phi \wedge M\neg\psi)$	**S4(3)** $\times$ **DR1** $\times$ 2
4.	$LM(L\phi \wedge M\neg\psi) \supset LML\phi \wedge LM\neg\psi$	**K3** $\times$ 3 $\times$ Eq
5.	$LML\phi \supset ML\phi$	**t**$[ML\phi/\phi]$
6.	$LML\phi \wedge LM\neg\psi \supset ML\phi \wedge LM\neg\psi$	P.C. $\times$ 5
7.	$LM(L\phi \wedge M\neg\psi) \supset ML\phi \wedge LM\neg\psi$	P.C. $\times$ 4 $\times$ 6
8.	$\neg ML\phi \vee \neg LM\neg\psi \supset \neg LM(L\phi \wedge M\neg\psi)$	P.C. $\times$ 7
9.	$\neg ML\phi \vee ML\psi \supset ML\neg(L\phi \wedge \neg L\psi)$	**K5** $\times$ 8
10.	$(ML\phi \supset ML\psi) \supset ML(L\phi \supset L\psi)$	P.C. $\times$ 9 □

Proof of (14)

$$
\begin{array}{lll}
 & \textit{not not not } \phi & \\
\equiv & L\neg LL\neg LL\neg L\phi & \\
\equiv & L\neg L\neg L\neg L\phi & \text{by } \mathbf{S4(2)} \\
\equiv & LML\neg L\phi & \\
\equiv & ML\neg L\phi & \text{by (4)} \\
\equiv & MLM\neg\phi & \\
\equiv & LM\neg\phi & (5) \\
\equiv & L\neg L\phi & \\
\equiv & \textit{not } \phi & \square
\end{array}
$$

Proof of (15)

$$
\begin{array}{lll}
 & \textit{not } (\phi, \psi) & \\
\equiv & \neg ML(\phi \wedge \psi) & \\
\equiv & \neg(ML\phi \wedge ML\psi) & \text{by (11)} \\
\equiv & \neg ML\phi \vee \neg ML\psi & \\
\equiv & \neg MML\phi \vee \neg MML\psi & \text{by } \mathbf{S4(3)} \\
\equiv & L\neg ML\phi \vee L\neg ML\psi & \\
\equiv & (\textit{not } \phi; \textit{not } \psi) & \square
\end{array}
$$

Proof of (16)

$$
\begin{array}{lll}
 & \textit{not } (\phi; \psi) & \\
\equiv & \neg ML(L\phi \vee L\psi) & \\
\equiv & \neg(ML\phi \vee ML\psi) & \text{by (12)} \\
\equiv & \neg ML\phi \wedge \neg ML\psi & \\
\equiv & (\textit{not } \phi, \textit{not } \psi) & \square
\end{array}
$$

APPENDIX B

Proofs of (meta)theorems in the paper

Proof of Theorem 1

- The condition $F \in H_\mathcal{M}$ holds iff F is true at any world $w \in W_1 \cup W_2$, which is equivalent to $\mathcal{M} \models LF$.
- For the left to right direction, if $F \in T_\mathcal{M}$ then all worlds in W_2 satisfy F. Therefore, for any $w \in W_2$, $(\mathcal{M}, w) \models LF$. Finally, as all worlds are connected to W_2 and W_2 is not empty, we have that, for any world $w \in W_1 \cup W_2$, $(\mathcal{M}, w) \models MLF$, that is, $\mathcal{M} \models MLF$. For the right to left direction, $\mathcal{M} \models MLF$ means that, at some world $w \in W_1 \cup W_2$, $(\mathcal{M}, w) \models LF$ must be true. If $w \in W_2$ then all worlds in W_2 satisfy F and so $F \in T_\mathcal{M}$. If $w \in W_1$, F is satisfied in any world, including cluster W_2, and we repeat the same argument. □

Lemma 2 *Let F be a classical formula, X a candidate set and $\mathcal{M}$ an S4F model such that $T_\mathcal{M} = X$. Then $\mathcal{M} \models (not\ F)^X$ iff $\mathcal{M} \models not\ F$.*

Proof

If directly follows from Theorem 1 and the definition of *not* F in IDL as $\neg MLF$. If any classical formula F belongs (resp. does not belong) to $T_\mathcal{M}$, we will have MLF true (resp. false) and so *not* F false (resp. true). □

Lemma 3 *Let $\mathcal{M} = \langle W_1, W_2, V\rangle$ be a S4F model, X a candidate set such that $H_\mathcal{M} = X$ and ϕ a formula in language $[;\ ,](\mathcal{L})$. Then, $\mathcal{M} \models L\phi$ iff $X \models_{ND} \phi$.*

Proof

We proceed by structural induction on ϕ.

1. For any classical formula F, $\mathcal{M} \models LF$ means that $F \in H_\mathcal{M} = X$, and so $X \models_{\text{ND}} F$.
2. We assume proved for α and β, subformulas of ϕ.
3. Assume $\phi = (\alpha, \beta)$, that is, $\phi = \alpha \wedge \beta$ after unfolding the definition of ',' in IDL. By (7), $\mathcal{M} \models L\alpha \wedge \beta$ corresponds to $\mathcal{M} \models L\alpha$ and $\mathcal{M} \models L\beta$. By induction hypothesis, this is the same than $X \models_{\text{ND}} \alpha$ and $X \models_{\text{ND}} \beta$, which corresponds to $X \models_{\text{ND}} (\alpha, \beta)$ in NDL.
4. Assume $\phi = (\alpha; \beta)$, that is, $\phi = L\alpha \vee L\beta$ after unfolding the definition of ';' in IDL. By (8), $\mathcal{M} \models L(L\alpha \vee L\beta)$ corresponds to $\mathcal{M} \models L\alpha$ or $\mathcal{M} \models L\beta$. By induction hypothesis, this is the same than $X \models_{\text{ND}} \alpha$ or $X \models_{\text{ND}} \beta$, which corresponds to $X \models_{\text{ND}} (\alpha; \beta)$ in NDL.

□

Proof of Theorem 2

By a simple adaptation of proofs in [10] for Nested Logic Programming, it is not difficult to show that equivalences (14)-(16) also hold for NDL, and we can assume that they have been previously applied on D. From Lemma 2 it is clear that $\mathcal{M} \models D$ iff $\mathcal{M} \models D^X$. Now, notice that for getting an extension X in NDL, it suffices with guaranteeing that X is minimal among those candidate sets $Y \models_{\text{ND}} D^X$ such that $Y \subseteq X$. Finally, as NDL formulas in D^X have the shape $[;\ ,](\mathcal{L})$, we can apply Lemma 3, to prove that any $Y \subseteq X$ satisfies $Y \models_{\text{ND}} D^X$ iff any S4F model $\mathcal{M}'$ with $H_{\mathcal{M}'} = Y$ and $T_{\mathcal{M}'} = X$ satisfies $\mathcal{M}' \models D^X$. □

Proof of Theorem 3

Like in the previous proof, we assume that default negations in D have been previously unfolded under equivalences (14)-(16). If so, Lemma 2 guarantees that $\mathcal{M} \models D$ iff $\mathcal{M} \models D^Y$. Now, for the left to right direction, assume that (X, Y) is an SE-model of D. Then, $X \models_{\text{ND}} D^Y$ implies, by Lemma 3, that $\mathcal{M} \models D^Y$, and so $\mathcal{M} \models D$.

For the right to left direction, assume $\mathcal{M} \models D$, and so $\mathcal{M} \models D^Y$. On the one hand, by Lemma 3 again, $X \models_{\text{ND}} D^Y$. On the other hand, by Property 1, $\mathcal{M} \models D^Y$ also implies $\langle W_2, V\rangle \models D^Y$. If we take $\mathcal{M}' = \langle W_2, W_2, V\rangle$, it is then clear that $\mathcal{M}' \models D^Y$. By construction $Y = T_\mathcal{M} = H_{\mathcal{M}'}$. Thus, we can apply Lemma 3 to conclude $Y \models_{\text{ND}} D^Y$. □

Lemma 4 *Let $\mathcal{M} = \langle W_1, W_2, V\rangle$ be a S4F model and $w \in W_1 \cup W_2$. Then the condition $(\mathcal{M}, w) \models L\phi$ is equivalent to:*

- *$\mathcal{M} \models \phi$, if $w \in W_1$,*
- *$\langle W_2, V\rangle \models \phi$, if $w \in W_2$.*

Proof

For the first case, any world $w \in W_1$ is connected to all worlds $W_1 \cup W_2$ in the model. Therefore, satisfaction of $L\phi$ in w means making ϕ true in all the worlds in $W_1 \cup W_2$, i.e., $\mathcal{M} \models \phi$.

For the second case, $(M, w) \models L\phi$ would mean that ϕ is true in all worlds in cluster W_2. But this is equivalent to assert that ϕ is true in all the worlds in the S4F model $\langle W_2, V\rangle$. □

Proof of Lemma 1

We begin observing that, when $W_1 = \emptyset$, we get $\mathcal{M} = \langle W_2, V\rangle$ and $\mathcal{I}^h_\mathcal{M} = \mathcal{I}^t_\mathcal{M}$ what, in fact, means that case (a) collapses into (b). Therefore, without loss of generality, we can assume that $W_1 \neq \emptyset$.

The proof proceeds by structural induction on ϕ.

1. Assume ϕ is some atom p. Then $(\mathcal{I}_\mathcal{M}, h) \models_{\text{HT}} p$ means that $p \in I^h_\mathcal{M}$. By definition of $I^h_\mathcal{M}$, this corresponds to $p \in I_w$, for all world $w \in W_1 \cup W_2$, which is equivalent to assert $\mathcal{M} \models p$. Similarly, $(\mathcal{I}_\mathcal{M}, t) \models_{\text{HT}} p$ means that $p \in I_w$, for all world $w \in W_2$, which is equivalent to $\langle W_2, V\rangle \models p$.
2. The induction hypothesis is that (a) and (b) hold, for any $\mathcal{M}$ and any subformula α (or β) of ϕ.
3. Assume $\phi = (\alpha, \beta)$. The modal translation of ϕ would simply be $\alpha \wedge \beta$. Then, for any S4F model $\mathcal{M}'$, we have the following equivalent conditions:

$$
\begin{array}{lll}
 & \mathcal{M}' \models (\alpha \wedge \beta) & \\
\text{iff} & \mathcal{M}' \models L(\alpha \wedge \beta) & \text{(Proposition 1)} \\
\text{iff} & \mathcal{M}' \models L\alpha \text{ and } \mathcal{M}' \models L\beta & \text{(by } \mathbf{K3}\text{)} \\
\text{iff} & \mathcal{M}' \models \alpha \text{ and } \mathcal{M}' \models \beta & \text{(Proposition 1)}
\end{array}
$$

To prove (a) we just take $\mathcal{M}' = \mathcal{M}$ and get the equivalent conditions:

$$\begin{array}{lll} & \mathcal{M} \models \alpha \text{ and } \mathcal{M} \models \beta & \\ \text{iff} & (\mathcal{I}_\mathcal{M}, h) \models_{\text{HT}} \alpha \text{and } (\mathcal{I}_\mathcal{M}, h) \models_{\text{HT}} \beta & \text{(induction hyp.)} \\ \text{iff} & (\mathcal{I}_\mathcal{M}, h) \models_{\text{HT}} \alpha \wedge \beta & \text{(HT satisfaction)} \end{array}$$

The proof for (b) is completely analogous, taking $\mathcal{M}' = \langle W_2, V \rangle$.

4. Assume $\phi = (\alpha; \beta)$. The modal translation of ϕ would be $L\alpha \vee L\beta$.

(a) Note that $\mathcal{M} \models L\alpha \vee L\beta$ is equivalent to:

$$\text{for all } w \in W_1 \cup W_2, \quad (\mathcal{M}, w) \models L\alpha \text{ or } (\mathcal{M}, w) \models L\beta$$

what can be separated into worlds in W_1 and in W_2:

$$\begin{array}{l} \text{for all } w \in W_1, \quad (\mathcal{M}, w) \models L\alpha \text{ or } (\mathcal{M}, w) \models L\beta \\ \quad \text{and} \\ \text{for all } w \in W_2, \quad (\mathcal{M}, w) \models L\alpha \text{ or } (\mathcal{M}, w) \models L\beta \end{array}$$

Now, applying Lemma 4, we get:

$$\begin{array}{l} \text{for all } w \in W_1, \quad \mathcal{M} \models \alpha \text{ or } \mathcal{M} \models \beta \\ \quad \text{and} \\ \text{for all } w \in W_2, \quad \langle W_2, V \rangle \models \alpha \text{ or } \langle W_2, V \rangle \models \beta \end{array}$$

As both W_1 and W_2 are not empty, we just get:

$$\begin{array}{l} \mathcal{M} \models \alpha \text{ or } \mathcal{M} \models \beta \\ \quad \text{and} \\ \langle W_2, V \rangle \models \alpha \text{ or } \langle W_2, V \rangle \models \beta \end{array}$$

but, due to Proposition 1, $\mathcal{M} \models \alpha$ implies $\langle W_2, V \rangle \models \alpha$ and the same holds for β. Thus, the condition we obtained just amounts to:

$$\mathcal{M} \models \alpha \text{ or } \mathcal{M} \models \beta$$

Finally, by induction hypothesis, the condition above is equivalent to:

$$(\mathcal{I}_\mathcal{M}, h) \models_{\text{HT}} \alpha \text{ or } (\mathcal{I}_\mathcal{M}, h) \models_{\text{HT}} \beta$$

which by definition of HT satisfaction is equivalent to $(\mathcal{I}_\mathcal{M}, h) \models_{\text{HT}} \alpha; \beta$.

(b) The condition $\langle W_2, V \rangle \models L\alpha \vee L\beta$ is equivalent to:

$$\text{for all } w \in W_2, \quad (\langle W_2, V \rangle, w) \models L\alpha \text{ or } (\langle W_2, V \rangle, w) \models L\beta$$

By Lemma 4 we get:

$$\text{for all } w \in W_2, \quad \langle W_2, V \rangle \models \alpha \text{ or } \langle W_2, V \rangle \models \beta$$

but as $W_2 \neq \emptyset$ this amounts to:

$$\langle W_2, V \rangle \models \alpha \text{ or } \langle W_2, V \rangle \models \beta$$

If we apply induction hypothesis, we obtain:

$$(\mathcal{I}_\mathcal{M}, t) \models_{\text{HT}} \beta \text{ or } (\mathcal{I}_\mathcal{M}, t) \models_{\text{HT}} \alpha$$

which by definition of HT satisfaction is equivalent to $(\mathcal{I}_\mathcal{M}, t) \models_{\text{HT}} \alpha; \beta$.

5. Assume $\phi = (\alpha \leftarrow \beta)$. The modal translation of ϕ is $L\beta \supset L\alpha$.

(a) Note that $\mathcal{M} \models L\beta \supset L\alpha$ is equivalent to:

$$\text{for all } w \in W_1 \cup W_2, \quad (\mathcal{M}, w) \not\models L\beta \text{ or } (\mathcal{M}, w) \models L\alpha$$

what can be separated again for worlds in W_1 and worlds in W_2:

$$\begin{array}{l} \text{for all } w \in W_1, \quad (\mathcal{M}, w) \not\models L\beta \text{ or } (\mathcal{M}, w) \models L\alpha \\ \quad \text{and} \\ \text{for all } w \in W_2, \quad (\mathcal{M}, w) \not\models L\beta \text{ or } (\mathcal{M}, w) \models L\alpha \end{array}$$

Now, applying Lemma 4, we get:

$$\begin{array}{l} \text{for all } w \in W_1, \quad \mathcal{M} \not\models \beta \text{ or } \mathcal{M} \models \alpha \\ \quad \text{and} \\ \text{for all } w \in W_2, \quad \langle W_2, V \rangle \not\models \beta \text{ or } \langle W_2, V \rangle \models \alpha \end{array}$$

As W_1 and W_2 are not empty, we just get:

$$\begin{array}{l} \mathcal{M} \not\models \beta \text{ or } \mathcal{M} \models \alpha \\ \quad \text{and} \\ \langle W_2, V \rangle \not\models \beta \text{ or } \langle W_2, V \rangle \models \alpha \end{array}$$

Finally, we apply induction hypothesis to the condition above, obtaining:

$$\begin{array}{l} (\mathcal{I}_\mathcal{M}, h) \not\models_{\text{HT}} \beta \text{ or } (\mathcal{I}_\mathcal{M}, h) \models_{\text{HT}} \alpha \\ \quad \text{and} \\ (\mathcal{I}_\mathcal{M}, t) \not\models_{\text{HT}} \beta \text{ or } (\mathcal{I}_\mathcal{M}, t) \models_{\text{HT}} \alpha \end{array}$$

which by definition of HT satisfaction is equivalent to $(\mathcal{I}_\mathcal{M}, h) \models_{\text{HT}} \alpha \leftarrow \beta$.

(b) We have now that $\langle W_2, V \rangle \models L\beta \supset L\alpha$ is equivalent to:

$$\text{for all } w \in W_2, \quad (\langle W_2, V \rangle, w) \not\models L\beta \text{ or } (\langle W_2, V \rangle, w) \models L\alpha$$

which by Lemma 4 and $W_2 \neq \emptyset$ amounts to:

$$\langle W_2, V \rangle \not\models \beta \text{ or } \langle W_2, V \rangle \models \alpha$$

Finally, applying induction hypothesis we get:

$$(\mathcal{I}_\mathcal{M}, t) \not\models_{\text{HT}} \beta \text{ or } (\mathcal{I}_\mathcal{M}, t) \models_{\text{HT}} \alpha$$

which by definition of HT satisfaction is equivalent to $(\mathcal{I}_\mathcal{M}, t) \models_{\text{HT}} \alpha \leftarrow \beta$.

6. The proof for $\phi = \textit{not } \alpha$ can be simply reduced to an instance of the previous case taking $\phi = \bot \leftarrow \alpha$, which is the definition of *not* α in HT. In the case of IDL, the formula $\bot \leftarrow \alpha$ would be translated as $L\alpha \supset L\bot$, that is, $\neg L\alpha$, whereas the translation of *not* α is $L\neg L\alpha$ as seen before. Although both formulas are not equivalent, Proposition 1 implies that: $\mathcal{M}' \models \neg L\alpha$ iff $\mathcal{M}' \models L\neg L\alpha$. Therefore, as the conditions (a) and (b) use this type of satisfaction (respectively taking $\mathcal{M}' = \mathcal{M}$ and $\mathcal{M}' = \langle W_2, V \rangle$) both formulas will behave in the same way and we can safely replace *not* α by $\bot \leftarrow \alpha$ in IDL.

□

Proof of Theorem 4

For the left to right direction, assume $\models_{\text{HT}} \phi$ but not $\models \phi$. Then, there exists some $\mathcal{M}$ such that $\mathcal{M} \not\models \phi$. But then, by Lemma 1, $(\mathcal{I}_{\mathcal{M}}, h) \not\models_{\text{HT}} \phi$ which contradicts the hypothesis $\models_{\text{HT}} \phi$.

For the right to left direction, assume $\models \phi$ but not $\models_{\text{HT}} \phi$, that is, $(\mathcal{I}, h) \not\models_{\text{HT}} \phi$, for some HT interpretation $\mathcal{I} = (I^h, I^t)$. We can construct $\mathcal{M} = \langle W_1, W_2, V \rangle$ such that, each world w is identified with its valuation I_w and:

$$\begin{array}{rcl} W_2 & = & \{I \subseteq At \mid I^t \subseteq I\} \\ W_1 & = & \{I \subseteq At \mid I^h \subseteq I\} - W_2 \end{array}$$

Then, $\mathcal{I}_{\mathcal{M}} = \mathcal{I}$ and, by Lemma 1, $\mathcal{M} \not\models \phi$, contradicting the hypothesis $\models \phi$. $\square$