# On the Logic and Computation of Partial Equilibrium Models (extended version)

Pedro Cabalar[1], Sergei Odintsov[2], David Pearce[3] and Agustín Valverde[4]*

[1] Corunna University (Corunna, Spain), cabalar@dc.fi.udc.es
[2] Sobolev Institute of Mathematics (Novosibirsk, Russia), odintsov@math.nsc.ru
[3] Universidad Rey Juan Carlos (Madrid, Spain), davidandrew.pearce@urjc.es
[4] University of Málaga (Málaga, Spain), a_valverde@ctima.uma.es

**Abstract.** The nonmonotonic formalism of *partial equilibrium logic* (PEL) was introduced and studied in [1,2] and proposed as a logical foundation for the partial stable and well-founded semantics of logic programs. Here we study further logical properties of PEL and some techniques to compute partial equilibrium models.

## 1 Introduction

Equilibrium logic [10] is a general nonmonotonic formalism based on the nonclassical logic $HT$ of *here-and-there*, (also known as Goedel's 3-valued logic). It was proposed in [9] as a logical foundation for the stable model semantics of logic programs. Recently in [1,2] *partial equilibrium logic* has been introduced and studied as a foundation for the partial stable (p-stable) and well-founded semantics (WFS) of logic programs. This formalism is based on a 6-valued logic which we denote by $HT^2$. Just as equilibrium models correspond to the stable models of programs as defined in [4], so partial equilibrium models correspond to the p-stable models defined in [13]. In each case the equilibrium construction is similarly based on taking certain total models that are minimal. The underlying models, however, are different in each case, and while total $HT$ models are complete in the sense of verifying either $\varphi$ or $\neg\varphi$ for any formula $\varphi$, the total $HT^2$ models are not.

The present paper continues the work of [1,2] whose main results were as follows: (i) partial equilibrium logic (PEL) was defined and p-equilibrium models were shown to coincide with p-stable models for logic programs.; (ii) the logic $HT^2$ was axiomatised and completeness shown; (iii) analogous to the case of equilibrium logic, it was shown that the strong equivalence of theories wrt PEL can be captured by equivalence in the logic $HT^2$; (iv) some properties of nonmonotonic entailment in PEL and its complexity were studied as well as a method for reducing PEL to ordinary equilibrium logic. Here we examine further logical and computational issues associated with p-equilibrium models and their underlying logics: $HT^2$ and the logic of total $HT^2$ models, which we denote by $HT^*$. In particular we provide in §4 a proof theory for PEL by presenting tableau calculi for the logics $HT$ and $HT^*$ as well as for p-equilibrium model checking. The calculus for $HT^2$ is of independent interest as a means for checking the strong

* This research was partially supported by CICyT project TIC-2003-9001-C02.

equivalence of theories. We also (§3) axiomatise the logic $HT^*$ and discuss its relation to other logics such as Przymusinski's $Prz_3$ [14]. Lastly in §5 we consider the method of *splitting* a logic program, a familiar technique for optimising computation under the stable model semantics [6,3]. We derive a splitting theorem for disjunctive and nested logic programs under PEL.

## 2 Logical preliminaries: the logics $HT^2$ and PEL

We introduce the logic $HT^2$ and its semantics, given in terms of $HT^2$ *frames*, and we define *partial equilibrium logic* (PEL) in terms of minimal $HT^2$ models. Formulas of $HT^2$ are built-up in the usual way using atoms from a given propositional signature $At$ and the standard logical constants: $\wedge$, $\vee$, $\rightarrow$, $\neg$. We write $\mathfrak{L}(At)$ to stand for the set of all well-formed formulae (ie, the language) under signature $At$. A set of $HT^2$ formulae is called a *theory*. The axiomatic system for $HT^2$ is described in two stages. In the first stage we include the following inference rules:

$$\frac{\alpha,\ \alpha \rightarrow \beta}{\beta} \text{ (Modus Ponens)} \qquad\qquad \frac{\alpha \rightarrow \beta}{\neg\beta \rightarrow \neg\alpha}$$

plus the axiom schemata of *positive logic* together with:

$$A1.\ \neg\alpha \wedge \neg\beta \rightarrow \neg(\alpha \vee \beta) \qquad A2.\ \neg(\alpha \rightarrow \alpha) \rightarrow \beta \qquad A3.\ \neg(\alpha \wedge \beta) \rightarrow \neg\alpha \vee \neg\beta$$

Thus, both De Morgan laws are provable in $HT^2$. Moreover, axiom A2 allows us to define intuitionistic negation, '$-$', in $HT^2$ as: $-\alpha := \alpha \rightarrow \neg(p_0 \rightarrow p_0)$.
In a second stage, we further include the rule $\frac{\alpha \vee (\beta \wedge \neg\beta)}{\alpha}$ and the axioms schemata:

A4. $-\alpha \vee - - \alpha$
A5. $-\alpha \vee (\alpha \rightarrow (\beta \vee (\beta \rightarrow (\gamma \vee -\gamma))))$
A6. $\bigwedge_{i=0}^{2}((\alpha_i \rightarrow \bigvee_{j \neq i} \alpha_j) \rightarrow \bigvee_{j \neq i} \alpha_j) \rightarrow \bigvee_{i=0}^{2} \alpha_i$
A7. $\alpha \rightarrow \neg\neg\alpha$
A8. $\alpha \wedge \neg\alpha \rightarrow \neg\beta \vee \neg\neg\beta$
A9. $\neg\alpha \wedge \neg(\alpha \rightarrow \beta) \rightarrow \neg\neg\alpha$
A10. $\neg\neg\alpha \vee \neg\neg\beta \vee \neg(\alpha \rightarrow \beta) \vee \neg\neg(\alpha \rightarrow \beta)$
A11. $\neg\neg\alpha \wedge \neg\neg\beta \rightarrow (\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$

$HT^2$ is determined by the above inference rules and the schemata A1-A11.

**Definition 1.** *A (Routley) frame is a triple $\langle W, \leq, * \rangle$, where $W$ is a set, $\leq$ a partial order on $W$ and $* : W \rightarrow W$ is such that $x \leq y$ iff $y^* \leq x^*$. A (Routley) model is a Routley frame together with a valuation $V$ ie. a function from $At \times W \longrightarrow \{0, 1\}$ satisfying (1): $V(p, u) = 1\ \ \&\ \ u \leq w \ \Rightarrow\ V(p, w) = 1$.*

The valuation $V$ is extended to all formulas via the usual rules for intuitionistic (Kripke) frames for the positive connectives $\wedge$, $\vee$, $\rightarrow$ where the latter is interpreted via the $\leq$ order:

$$V(\varphi \rightarrow \psi, w) = 1 \text{ iff for all } w' \text{ such that } w \leq w', \quad V(\varphi, w') = 1 \Rightarrow V(\psi, w') = 1$$

The main difference with respect to intuitionistic frames is the presence of the $*$ operator that is used for interpreting negation via the following condition:

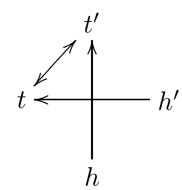
$$V(\neg\varphi, w) = 1 \ \text{ iff } \ \ V(\varphi, w^*) = 0.$$

A proposition $\varphi$ is said to be *true* in a model $\mathcal{M} = \langle W, \leq, *, V\rangle$, if $V(\varphi, v) = 1$, for all $v \in W$. A formula $\varphi$ is *valid*, in symbols $\models \varphi$, if it is true in every model. It is easy to prove by induction that condition (1) in Definition 1 above holds for any formula $\varphi$, ie

$$V(\varphi, u) = 1 \ \ \& \ \ u \leq w \Rightarrow V(\varphi, w) = 1. \tag{1}$$

**Definition 2** **($HT^2$ model).** *An* $HT^2$ model *is a Routley model* $\mathcal{M} = \langle W, \leq, R, V\rangle$ *such that (i)* $W$ *comprises 4 worlds denoted by* $h, h', t, t'$*, (ii)* $\leq$ *is a partial ordering on* $W$ *satisfying* $h \leq t$*,* $h \leq h'$*,* $h' \leq t'$ *and* $t \leq t'$*, (iii) the* $*$ *operation is determined by* $h^* = t^* = t'$*,* $(h')^* = (t')^* = t$*, (iv)* $V$ *is a-valuation.*

The diagram on the right depicts the $\leq$-ordering among worlds (a strictly higher location means $\geq$) and the action of the $*$- mapping using arrows.




Truth and validity for $HT^2$ models are defined analogously to the previous case and from now on we let $\models$ denote the truth (validity) relation for $HT^2$ models. We have the following completeness theorem[5]:

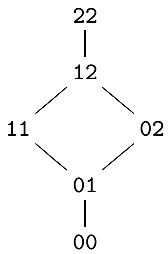
**Theorem 1** **([1]).** $\models \varphi$ *iff* $\varphi$ *is a theorem of* $HT^2$*.*

## 2.1 $HT^2$ as a 6-valued logic

Now, consider an $HT^2$ model $\mathcal{M} = \langle W, \leq, ^*, V\rangle$ and let us denote by $H, H', T, T'$ the four sets of atoms respectively verified at each corresponding point or world $h, h', t, t'$. More succinctly, we can represent $\mathcal{M}$ as the pair $\langle \mathbf{H}, \mathbf{T}\rangle$ so that we group each pair of unprimed/primed world as $\mathbf{H} = (H, H')$ and $\mathbf{T} = (T, T')$. By construction, each of these pairs $\mathbf{I} = (I, I')$ satisfies $I \subseteq I'$, so that $\mathbf{I}$ can be seen as a 3-valued interpretation. Given $\mathbf{I}$ and an atom $p$, we use the values $\{0, 1, 2\}$ to respectively denote $p \in I$, $p \in I' \setminus I$ and $p \notin I'$. As we have two pairs like this, $\langle \mathbf{H}, \mathbf{T}\rangle$, the possible "situations" of a formula in $HT^2$ can be defined by a pair of values $xy$ with $x, y \in \{0, 1, 2\}$. Condition (1) restricts the number of these situations to the following six $\mathtt{00} := \emptyset$, $\mathtt{01} := \{t'\}$, $\mathtt{11} := \{h', t'\}$, $\mathtt{02} := \{t, t'\}$, $\mathtt{12} := \{h', t, t'\}$, $\mathtt{22} := W$ where each set shows the worlds at which the formula is satisfied. Thus, an alternative way of describing $HT^2$ is by providing its logical matrix in terms of a 6-valued logic. As a result, the above setting becomes an algebra of 6 cones: $\mathcal{A}^{HT^2} := \langle\{\mathtt{00}, \mathtt{01}, \mathtt{11}, \mathtt{02}, \mathtt{12}, \mathtt{22}\}, \vee, \wedge, \rightarrow, \neg\rangle$ where $\vee$ and $\wedge$ are set theoretical join and meet, whereas $\rightarrow$ and $\neg$ are defined as follows: $x \rightarrow y := \{w : w \leq w' \Rightarrow (w' \in x \Rightarrow w' \in y)\}$, $\quad \neg\, x := \{w : w^* \notin x\}$.

The only distinguished element is $\mathtt{22}$. The lattice structure of this algebra can be described by the condition $xy \leq zt \Leftrightarrow x \leq z \ \& \ y \leq t$ and is shown in Figure 1, together with the resulting truth-tables.

[5] The first stage alone defines a logic complete for the general Routley frames.




| $\varphi$ | $\neg\varphi$ |
|---|---|
| 00 | 22 |
| 01 | 11 |
| 11 | 11 |
| 02 | 00 |
| 12 | 00 |
| 22 | 00 |

| $\rightarrow$ | 00 | 01 | 11 | 02 | 12 | 22 |
|---|---|---|---|---|---|---|
| 00 | 22 | 22 | 22 | 22 | 22 | 22 |
| 01 | 00 | 22 | 22 | 22 | 22 | 22 |
| 11 | 00 | 02 | 22 | 02 | 22 | 22 |
| 02 | 00 | 11 | 11 | 22 | 22 | 22 |
| 12 | 00 | 01 | 11 | 02 | 22 | 22 |
| 22 | 00 | 01 | 11 | 02 | 12 | 22 |

$V(\phi \wedge \psi) = \min\{V(\phi), V(\psi)\}, V(\phi \vee \psi) = \max\{V(\phi), V(\psi)\}$

**Fig. 1.** Lattice structure and truth tables for the 6-valued $HT^2$ description.

### 2.2 minimal models and relation to logic programs

The truth-ordering relation among 3-valued interpretations $\mathbf{I}_1 \leq \mathbf{I}_2$ is defined so that $\mathbf{I}_1$ contains less true atoms and more false ones (wrt set inclusion) than $\mathbf{I}_2$. Note that by the semantics, if $\langle \mathbf{H}, \mathbf{T} \rangle$ is a model then necessarily $\mathbf{H} \leq \mathbf{T}$, since it is easy to check that this condition is equivalent to $H \subseteq T$ and $H' \subseteq T'$. Moreover, for any theory $\Pi$ note that if $\langle \mathbf{H}, \mathbf{T} \rangle \models \Pi$ then also $\langle \mathbf{T}, \mathbf{T} \rangle \models \Pi$.

The ordering $\leq$ is extended to a partial ordering $\trianglelefteq$ among models as follows. We set $\langle \mathbf{H}_1, \mathbf{T}_1 \rangle \trianglelefteq \langle \mathbf{H}_2, \mathbf{T}_2 \rangle$ if (i) $\mathbf{T}_1 = \mathbf{T}_2$; (ii) $\mathbf{H}_1 \leq \mathbf{H}_2$. A model $\langle \mathbf{H}, \mathbf{T} \rangle$ in which $\mathbf{H} = \mathbf{T}$ is said to be *total*. Note that the term *total* model does not refer to the absence of undefined atoms. To represent this, we further say that a total partial equilibrium model is *complete* if $\mathbf{T}$ has the form $(T, T)$.

We are interested here in a special kind of minimal model that we call a partial equilibrium (or p-equilibrium) model. Let $\Pi$ be a theory.

**Definition 3 (Partial equilibrium model).** *A model $\mathcal{M}$ of $\Pi$ is said to be a* partial equilibrium *model of $\Pi$ if (i) $\mathcal{M}$ is total; (ii) $\mathcal{M}$ is minimal among models of $\Pi$ under the ordering $\trianglelefteq$.*

In other words a p-equilibrium model of $\Pi$ has the form $\langle \mathbf{T}, \mathbf{T} \rangle$ and is such that if $\langle \mathbf{H}, \mathbf{T} \rangle$ is any model of $\Pi$ with $\mathbf{H} \leq \mathbf{T}$, then $\mathbf{H} = \mathbf{T}$. We will sometimes use the abbreviation $\mathbf{T} \models\!\approx \Pi$ to denote that $\langle \mathbf{T}, \mathbf{T} \rangle$ is a p-equilibirum model of theory $\Pi$. *Partial equilibrium logic* (PEL) is the logic determined by truth in all p-equilibrium models of a theory.

We turn to the relation between PEL and logic programs. A *disjunctive logic program* is a set of formulas (also called *rules*) of the form

$$a_1 \wedge \ldots \wedge a_m \wedge \neg b_1 \wedge \ldots \wedge \neg b_n \rightarrow c_1 \vee \ldots \vee c_k \qquad (2)$$

where the $a, b, c$ with subscripts range over atoms and $m, n, k \geq 0$; for the definition of the p-stable models of a disjunctive logic program $\Pi$, see [13].

**Theorem 2 ([2]).** *A total $HT^2$ model $\langle \mathbf{T}, \mathbf{T} \rangle$ is a p-equilibrium model of a disjunctive program $\Pi$ iff the 3-valued interpretation $\mathbf{T}$ is a p-stable model of $\Pi$.*

We define a further partial ordering on total models by $\langle \mathbf{T}_1, \mathbf{T}_1 \rangle \preceq \langle \mathbf{T}_2, \mathbf{T}_2 \rangle$ if both $T_1 \subseteq T_2$ and $T_2' \subseteq T_1'$. Then we say that a total $HT^2$ model that is $\preceq$-minimal among the p-equilibrium models of a theory $\Gamma$ is a *well-founded model* of $\Gamma$. This terminology is justified by the fact that if $\Pi$ is a normal logic program, the unique $\preceq$-minimal p-equilibrium model of $\Pi$ coincides with the well-founded model of $\Pi$ in the sense of [17].

The notion of strong equivalence for logic programs was introduced in [7] and logically characterised for the case of programs under stable model semantics. The study of strong equivalence, its generalisations and computation, has since become a lively research area, with potential for application to program optimisation. Until now there was no analogous research programme for p-stable and WF semantics. A basis is provided however by Theorem 3 below and several extensions proved in [2].

**Definition 4 ((strongly) equivalent theories).** *Two theories $\Pi, \Pi'$ are said to be (PEL)-equivalent or simply* equivalent *(resp.* strongly equivalent*), in symbols $\Pi \equiv \Pi'$ (resp. $\Pi \equiv_s \Pi'$), iff they have the same p-equilibrium models (resp. iff for any $\Gamma$, $\Pi \cup \Gamma \equiv \Pi' \cup \Gamma$).*

**Theorem 3 ([1]).** *Two theories $\Pi, \Pi'$ are strongly equivalent iff they are $HT^2$ equivalent, ie have the same $HT^2$ models.*

This provides added interest in computational proof systems for $HT^2$.

### 2.3 Complexity of reasoning in $HT^2$ and PEL

We denote by $SAT_{CL}$ and $VAL_{CL}$ the classes of satisfiable formulas and valid formulas respectively in classical logic, and $SAT_{HT^2}$ and $VAL_{HT^2}$ the classes of satisfiable formulas and valid formulas respectively in $HT^2$ logic.

**Theorem 4 ([2]).** *(i) $SAT_{HT^2}$ is NP-complete and $VAL_{HT^2}$ is coNP-complete; (ii) the problem of deciding whether a formula in $HT^2$ has partial equilibrium models is $\Sigma_2^P$-hard.*

**Corollary 1 ([2]).** *(i) The problem of checking the strong equivalence of theories is coNP-complete. (ii) The decision problem for equilibrium entailment is $\Pi_2^P$-hard.*

## 3 The Logic of Total Models

Total models play an important role in the definition of PEL since p-equilibrium models are a special kind of total model. We describe the logic of total models.

First note that total models can be distinguished among all $HT^2$-models via the scheme $\neg\neg\varphi \rightarrow \varphi$. For an $HT^2$ model $\mathcal{M} = \langle (H, H'), (T, T') \rangle = \langle \mathcal{W}^{\mathcal{HT}^{\in}}, \mathcal{V} \rangle$, set

$$\Delta_w^{\mathcal{M}} := \{\varphi : V(\varphi, w) = 1\}$$

for $w \in W^{HT^2}$. Obviously, $H \supset \Delta_h^{\mathcal{M}}$, $H' \supset \Delta_{h'}^{\mathcal{M}}$, etc. We omit the superscript $\mathcal{M}$ if it does not lead to confusion.

**Proposition 1.** *The following items are equivalent:*

1. $\langle \mathbf{H}, \mathbf{T} \rangle \models \neg\neg\varphi \to \varphi$ *for any* $\varphi$*,*
2. $\mathbf{H} = \mathbf{T}$*,*
3. $\Delta_h = \Delta_t$ *and* $\Delta_{h'} = \Delta_{t'}$*.*

Let us set $HT^* := HT^2 + \{\neg\neg p \to p\}$. From the last proposition, it follows that the number of possible situations of a formula in a total $HT^2$-model is reduced to the following three, $\mathtt{00} := \emptyset,\ \mathtt{11} := \{h', t'\},\ \mathtt{22} := \{h, h', t, t'\},$ where each set shows the worlds at which the formula is satisfied. Thus, logic $HT^*$ can be characterized by the three-element algebra: $\mathcal{A}^{HT^*} := \langle \{\mathtt{00}, \mathtt{11}, \mathtt{22}\}, \vee, \wedge, \to, \neg \rangle$ with the only distinguished element $\mathtt{22}$ and operations determined as the restrictions of the respective operation of the algebra $\mathcal{A}^{HT^2}$. It is routine to check that the set $\{\mathtt{00}, \mathtt{11}, \mathtt{22}\}$ is closed under $\mathcal{A}^{HT^2}$-operations.

At the same time, $HT^*$ differs from Przymusinski's logic $Prz_3$ [14] as well as from $\mathbf{N}_3$ [16,11], classical explosive logic with strong negation. All these logics are three-valued and the operations $\vee$ and $\wedge$ determine the structure of a linearly ordered lattice on the set of truth-values. If we denote the least truth-value in all these logics by $\mathtt{00}$, the greatest by $\mathtt{22}$, and the intermediate by $\mathtt{11}$, we see that all the logics have the same connectives $\neg$, $\vee$, $\wedge$, but different implications:

| $\to_{HT^*}$ | 00 | 11 | 22 |
|---|---|---|---|
| 00 | 22 | 22 | 22 |
| 11 | 00 | 22 | 22 |
| 22 | 00 | 11 | 22 |

| $\to_{\mathbf{N}_3}$ | 00 | 11 | 22 |
|---|---|---|---|
| 00 | 22 | 22 | 22 |
| 11 | 22 | 22 | 22 |
| 22 | 00 | 11 | 22 |

| $\to_{Prz_3}$ | 00 | 11 | 22 |
|---|---|---|---|
| 00 | 22 | 22 | 22 |
| 11 | 00 | 22 | 22 |
| 22 | 00 | 00 | 22 |

Comparing $HT^*$ and $\mathbf{N}_3$ we note the following

**Proposition 2.** $HT^* \subsetneq \mathbf{N}_3$, $\neg(p \to q) \leftrightarrow (p \wedge \neg q) \notin HT^*$.

For the comparison of $HT^*$ and $Prz_3$, recall that the language of $Prz_3$ contains also the necessity operator $l$ ($l\mathtt{22} = \mathtt{22}$, $lx = \mathtt{00}$ otherwise) and $\to_{Prz_3}$ can be defined via $\neg$, $\vee$, $\wedge$ and $l$: $\quad \varphi \to_{Prz_3} \psi := (\neg l\varphi \vee l\psi) \wedge (\neg l \neg\psi \vee l\neg\varphi)$.

At the same time, $l\varphi$ can be defined via implication as $\top \to_{Prz_3} \varphi$. It is well known that the operator $l$ is not definable in Łukasiewcz's three valued logic $Ł_3$ and that $Ł_3$ is equivalent to $\mathbf{N}_3$. Therefore, the operators $l$ and $\to_{Prz_3}$ are not definable in $\mathbf{N}_3$. Consequently, $l$ and $\to_{Prz_3}$ are not definable in the weaker logic $HT^*$.

**Proposition 3.** *Logic* $Prz_3$ *is not definable in* $HT^*$*.*

A simple axiomatisation of $HT^*$ modulo the basic logic $N^*$ is given by the following

**Proposition 4.** $HT^* = N^* + \{p \vee (p \to q) \vee -q,\ p \leftrightarrow \neg\neg p,\ p \wedge \neg p \to q \vee \neg q\}$.

*Proof.* In fact, the proof of these statement is a simplified version of the completeness proof for $HT^2$ in [1].

Thus, we obtain $HT^*$ by extending the intuitionistic fragment to $HT$ and adding the elimination of double negation and the Kleene axiom. Despite the fact that $HT^*$ and $HT$ have the same intuitionistic fragment, they have different negations and $HT^* \neq HT$. We can obtain $HT$ from $HT^2$ in the following way.

**Proposition 5.** *The addition to $HT^2$ of axiom (I) $= \neg\varphi \wedge \varphi \rightarrow \bot$, is equivalent to the condition $T = T'$.* □

**Proposition 6.** *The addition to $HT^2$ of De Jongh and Hendrik's axiom (used to obtain $HT$ from intuitionistic logic), (dJH) $= \varphi \vee (\varphi \rightarrow \psi) \vee -\varphi$ is equivalent to the condition: $T, H' \in \{H, T'\}$.*

**Proposition 7 (reduction to $HT$).** $HT = HT^2 \cup$ *(I)* $\cup$ *(dJH).*

## 4 A Tableau Calculus for PEL

We can describe a tableaux system for $HT^2$ using the standard methods for finite-valued logics [5,11]. The formulas in the tableau nodes are labelled with a set of truth-values, named *signs*, and these signs are propagated to the subformulas using the expansion rules. The family of the signs depends on the logic in question and it is possible to describe several tableaux systems for the same logic. For $HT^2$ we will use the following signs, where $[\geq v] = \{w \in \mathbf{6} \mid w \geq v\}$, and $[\leq v] = \{w \in \mathbf{6} \mid w \leq v\}$:

$$\{00\}, \{01\}, \{11\}, \{02\}, \{22\}, \{01, 11\}, [\leq 01], [\leq 11], [\leq 12], [\geq 01], [\geq 02], [\geq 12]$$

The usual notions of *closed* and *terminated* tableaux can be used in different ways. In the following definition we introduce the concept of *closed tableau* in order to characterise validity in $HT^2$.

**Definition 5.** *Let $\varphi$ be a formula in $HT^2$:*

1. *The* Initial tableau *to check the validity of $\varphi$ is: $T_0 = {\scriptstyle [\leq 12]:}\varphi$*
2. *If $T$ is a tableau and $T'$ is the tree obtained from $T$ applying one of the expansion rules in figure 2, then $T'$ is tableau for $\varphi$.*
3. *A branch $B$ in a tableau $T$ is called* closed *if one of the following condition hold: (i) it contains the constant $\bot$; (ii) it contains* signed literals, *${\scriptstyle S_1:}p, \ldots, {\scriptstyle S_n:}p$, such that $\cap_{i=1}^{n} S_i = \varnothing$. A tableau $T$ is called* closed *if every branch is closed.*

Intuitively, with the initial tableau ${\scriptstyle [\leq 12]:}\varphi$ we ask if it is possible to find an assignment for $\varphi$ that evaluates in $[\leq 12]$, in other words a countermodel. The expansion rules search for ways to evaluate the subformulas so as to define the countermodel.

**Theorem 5 (Soundness and completeness of the tableaux system).** *The formula $\varphi$ is valid in $HT^2$ if and only if there exists a closed tableau for it.*

### 4.1 Partial equilibrium models

Tableaux systems can also be used to study additional properties and relations [11,12]. In this section we define a system based con auxiliary tableaux in order to generate the partial equilibrium models of a theory. We proceed in two phases. First, we generate the total models of a theory by means of a tableau system in which we search for a *terminted tableau*. Then, for every total model, an auxiliary tableau is constructed to check whether the model in question is in partial equilibrium or not.

$$\frac{\{22\}{:}\varphi\to\psi}{\{00\}{:}\varphi \mid \{22\}{:}\psi \mid [\le 01]{:}\varphi,\ [\ge 01]{:}\psi \mid [\le 12]{:}\varphi,\ [\ge 12]{:}\psi \mid \{11\}{:}\varphi,\ \{11\}{:}\psi \mid \{02\}{:}\varphi,\ \{02\}{:}\psi}\qquad \frac{\{00\}{:}\varphi\to\psi}{[\ge 01]{:}\varphi,\ \{00\}{:}\psi}\qquad \frac{[\le 01]{:}\varphi\to\psi}{[\ge 01]{:}\varphi,\ \{00\}{:}\psi \mid [\ge 12]{:}\varphi,\ [\le 01]{:}\psi}$$

$$\frac{[\ge 01]{:}\varphi\to\psi}{\{22\}{:}\varphi \mid [\ge 01]{:}\psi}\qquad \frac{[\le 12{:}\varphi\to\psi}{[\ge 01]{:}\varphi,\ \{00\}{:}\psi \mid \{11\}{:}\varphi,\ [\le 01]{:}\psi \mid \{11\}{:}\varphi,\ \{02\}{:}\psi \mid \{02\}{:}\varphi,\ \{01,11\}{:}\psi \mid [\ge 12]{:}\varphi,\ [\le 11]{:}\psi \mid [\ge 12]{:}\varphi,\ \{02\}{:}\psi \mid \{22\}{:}\varphi,\ [\le 12]{:}\psi}$$

$$\frac{[\ge 12]{:}\varphi\to\psi}{\{00\}{:}\varphi \mid \ge 12{:}\psi \mid [\le 01]{:}\varphi,\ [\ge 01]{:}\psi \mid \{11\}{:}\varphi,\ \{11\}{:}\psi \mid \{02\}{:}\varphi,\ \{02\}{:}\psi}\qquad \frac{\{11\}{:}\varphi\to\psi}{\{02\}{:}\varphi,\ \{01,11\}{:}\psi \mid [\ge 02]{:}\varphi,\ \{11\}{:}\psi}\qquad \frac{\{02\}{:}\varphi\to\psi}{\{11\}{:}\varphi,\ \{01\}{:}\psi \mid \{11\}{:}\varphi,\ \{02\}{:}\psi \mid [\ge 12]{:}\varphi,\ \{02\}{:}\psi}$$

$$\frac{\{01,11\}{:}\varphi\to\psi}{[\ge 02]{:}\varphi,\ \{01,11\}{:}\psi}\qquad \frac{\{01\}{:}\varphi\to\psi}{[\ge 12]{:}\varphi,\ \{01\}{:}\psi}\qquad \frac{[\le 11]{:}\varphi\to\psi}{[\ge 01]{:}\varphi,\ \{00\}{:}\psi \mid [\ge 02]{:}\varphi,\ [\le 11]{:}\psi}\qquad \frac{[\ge 02]{:}\varphi\to\psi}{\{22\}{:}\varphi \mid [\ge 02]{:}\psi \mid \{01,11\}{:}\varphi,\ \{01,11\}{:}\psi}$$

$$\frac{\{01\}{:}\neg\varphi}{\bot}\qquad \frac{\{02\}{:}\neg\varphi}{\bot}\qquad \frac{\{22\}{:}\neg\varphi}{\{00\}{:}\varphi}\qquad \frac{[\ge 12]{:}\neg\varphi}{\{00\}{:}\varphi}\qquad \frac{[\ge 02]{:}\neg\varphi}{\{00\}{:}\varphi}\qquad \frac{\{00\}{:}\neg\varphi}{[\ge 02]{:}\varphi}$$

$$\frac{[\le 01]{:}\neg\varphi}{[\ge 02]{:}\varphi}\qquad \frac{[\le 11]{:}\neg\varphi}{[\ge 01]{:}\varphi}\qquad \frac{[\le 12]{:}\neg\varphi}{[\ge 01]{:}\varphi}\qquad \frac{[\ge 01]{:}\neg\varphi}{[\le 11]{:}\varphi}\qquad \frac{\{11\}{:}\neg\varphi}{\{01,11\}{:}\varphi}\qquad \frac{\{01,11\}{:}\neg\varphi}{\{01,11\}{:}\varphi}$$

For $v \in \{00, 11, 12\}$: $\frac{[\le v]{:}\varphi\wedge\psi}{[\le v]{:}\varphi \mid [\le v]{:}\psi}$; for $v \in \{01, 02, 12, 22\}$: $\frac{[\ge v]{:}\varphi\wedge\psi}{[\ge v]{:}\varphi,\ [\ge v]{:}\psi}$

$$\frac{\{01\}{:}\varphi\wedge\psi}{\{01\}{:}\varphi,\ [\ge 01]{:}\psi \mid [\ge 01]{:}\varphi,\ \{01\}{:}\psi \mid \{11\}{:}\varphi,\ \{02\}{:}\psi \mid \{02\}{:}\varphi,\ \{11\}{:}\psi}\qquad \frac{\{11\}{:}\varphi\wedge\psi}{\{11\}{:}\varphi,\ \{11\}{:}\psi \mid \{11\}{:}\varphi,\ [\ge 12]{:}\psi \mid [\ge 12]{:}\varphi,\ \{11\}{:}\psi}\qquad \frac{\{02\}{:}\varphi\wedge\psi}{\{02\}{:}\varphi,\ [\ge 02]{:}\psi \mid [\ge 02]{:}\varphi,\ \{02\}{:}\psi}$$

$$\frac{\{01,11\}{:}\varphi\wedge\psi}{\{01,11\}{:}\varphi,\ [\ge 01]{:}\psi \mid [\ge 01]{:}\varphi,\ \{01,11\}{:}\psi}\qquad \frac{[\le 01]{:}\varphi\wedge\psi}{[\le 01]{:}\varphi \mid [\le 01]{:}\psi \mid \{11\}{:}\varphi,\ \{02\}{:}\psi \mid \{02\}{:}\varphi,\ \{11\}{:}\psi}$$

For $v \in \{00, 01, 11, 12\}$: $\frac{[\le v]{:}\varphi\vee\psi}{[\le v]{:}\varphi,\ [\le v]{:}\psi}$; for $v \in \{01, 02, 22\}$: $\frac{[\ge v]{:}\varphi\vee\psi}{[\ge v]{:}\varphi \mid [\ge v]{:}\psi}$

$$\frac{\{01\}{:}\varphi\vee\psi}{[\le 01]{:}\varphi,\ \{01\}{:}\psi \mid \{01\}{:}\varphi,\ [\le 01]{:}\psi}\qquad \frac{\{11\}{:}\varphi\vee\psi}{[\le 11]{:}\varphi,\ \{11\}{:}\psi \mid \{11\}{:}\varphi,\ [\le 11]{:}\psi}\qquad \frac{\{02\}{:}\varphi\vee\psi}{[\le 01]{:}\varphi,\ \{02\}{:}\psi \mid \{02\}{:}\varphi,\ [\le 01]{:}\psi \mid \{02\}{:}\varphi,\ \{02\}{:}\psi}$$

$$\frac{\{01,11\}{:}\varphi\vee\psi}{\{01,11\}{:}\varphi,\ [\le 11]{:}\psi \mid [\le 11]{:}\varphi,\ \{01,11\}{:}\psi}\qquad \frac{[\ge 12]{:}\varphi\vee\psi}{[\ge 12]{:}\varphi \mid [\ge 12]{:}\psi \mid \{11\}{:}\varphi,\ \{02\}{:}\psi \mid \{02\}{:}\varphi,\ \{11\}{:}\psi}$$

**Fig. 2.** Expansion rules for $HT^2$

$$\frac{\{22\}{:}\varphi\to\psi}{\{00\}{:}\varphi \mid \{22\}{:}\psi \mid [\le 11]{:}\varphi,\ [\ge 11]{:}\psi}\qquad \frac{[\ge 11]{:}\varphi\to\psi}{\{00\}{:}\varphi \mid [\ge 11]{:}\psi}\qquad \frac{[\le 11]{:}\varphi\to\psi}{[\ge 11]{:}\varphi,\ \{22\}{:}\psi \mid \{22\}{:}\varphi,\ [\le 11]{:}\psi}\qquad \frac{\{00\}{:}\varphi\to\psi}{[\ge 11]{:}\varphi,\ \{00\}{:}\psi}$$

$$\frac{[\ge 11]{:}\neg\varphi}{[\le 11]{:}\varphi}\qquad \frac{[\le 11]{:}\neg\varphi}{[\ge 11]{:}\varphi}\qquad \frac{\{00\}{:}\neg\varphi}{\{22\}{:}\varphi}\qquad \frac{\{22\}{:}\neg\varphi}{\{00\}{:}\varphi}$$

For $v \in \{00, 11, 22\}$:

$$\frac{[\le v]{:}\varphi\wedge\psi}{[\le v]{:}\varphi \mid [\le v]{:}\psi}\qquad \frac{[\ge v]{:}\varphi\wedge\psi}{[\ge v]{:}\varphi,\ [\ge v]{:}\psi}\qquad \frac{[\le v]{:}\varphi\vee\psi}{[\le v]{:}\varphi,\ [\le v]{:}\psi}\qquad \frac{[\ge v]{:}\varphi\vee\psi}{[\ge v]{:}\varphi \mid [\ge v]{:}\psi}$$

**Fig. 3.** Expansion rules for total models of $HT^2$, ie. for $HT^*$

The total assignments evaluate formulas in $\{00, 11, 22\}$ and thus we only need to work with the following system of signs: $[\leq 11] = \{00, 11\}$, $[\leq 00] = \{00\}$, $[\geq 11] = \{11, 22\}$, $[\geq 11] = \{22\}$.

**Definition 6.** *Let $\Pi = \{\varphi_1, \ldots, \varphi_n\}$ a theory in $HT^2$:*

1. *The* Initial tableau *to generate total models is a single branch tree containing the following signed formulas:* ${\scriptstyle\{22\}}{:}\varphi_1, \ldots, {\scriptstyle\{22\}}{:}\varphi_n$.
2. *If $T$ is a tableau and $T'$ is the tree obtained from $T$ by applying one of the expansion rules in figure 3, then $T'$ is tableau for $\varphi$. As usual in tableaux systems for propositional logics, if a formula can be used to expand the tableau, then the tableau is expanded in every branch below the formula using the corresponding rule, and the formula used to expand is marked and is no longer used.*
3. *A branch in a tableau $T$ is called* closed *if the* signed literals *for a variable $p$, $S_1{:}p, \ldots, S_m{:}p$, verify $\cap_{i=1}^n S_i = \varnothing$. It is call* open *otherwise.*
4. *A branch in a tableau $T$ is called* finished *if it doesn't contain non-marked formulas.*
5. *A tableau $T$ is called* closed *if every branch is closed, and it is* terminated *if every branch is either closed or finished.*

In this case the tableau begins with formulas signed with 22, since we are looking for models. The expansion rules guarantee the construction of all possible models in such a way that when all formulas have been expanded, all the models can be determined on the basis of open branches.

**Theorem 6.** *Let $T$ be a non-closed terminated tableau for $\Pi$, and let $\{S_1{:}p_1, \ldots, S_n{:}p_n\}$ be the set of signed literals in an open branch. Then every assignment $V$ verifying $V(p_i) \in S_i$, for all $i$, is a total model of $\varphi$. Moreover, all the total models of $\Pi$ are generated from $T$ in this way.*

Example: (Taken from [2]) The figure 4 shows the tableau for the theory $\Pi = \{\neg p \to q \vee r, p \vee r\}$

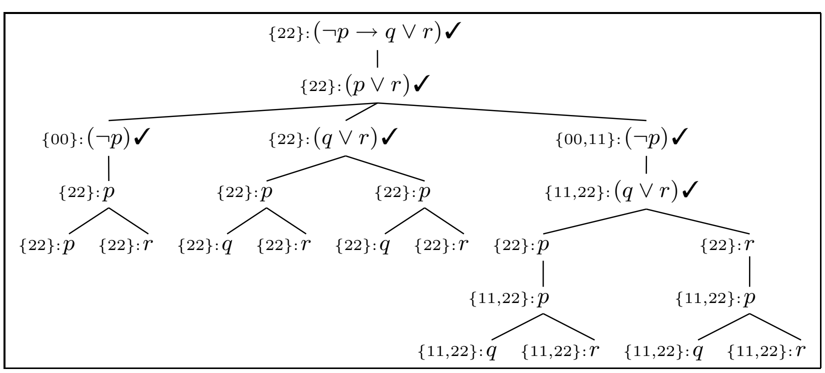



**Fig. 4.**

The tableau is finished and allows us to construct the set of total models of $\Pi$, as shown in the following table:

| | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ | $\sigma_5$ | $\sigma_6$ | $\sigma_7$ | $\sigma_8$ | $\sigma_9$ | $\sigma_{10}$ | $\sigma_{11}$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{14}$ | $\sigma_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 11 | 11 | 11 | 00 | 00 | 00 |
| $q$ | 22 | 22 | 22 | 11 | 11 | 11 | 00 | 00 | 00 | 22 | 11 | 00 | 22 | 11 | 00 |
| $r$ | 22 | 11 | 00 | 22 | 11 | 00 | 22 | 11 | 00 | 22 | 22 | 22 | 22 | 22 | 22 |

**Auxiliary tableau to check the partial equilibrium property** A total model is in partial equilibrium if there is no other model of the theory less than it under the partial ordering $\lhd$. In terms of the many-valued semantics, this ordering is defined between assignments based on the following relations between truth-values: $01 \lhd 11$, $02 \lhd 12 \lhd 22$. To look for such a model we construct an initial tableau specifically for each total model by applying the expansion rules in figure 2.

**Definition 7.** *Let $\varphi$ be a formula in $HT^2$ and $V$ a total model of it.*

1. *The* Initial tableau *to check the partial equilibrium property of $V$ for $\varphi$ is a single branch tree containing the following signed formulas: ${\scriptstyle \{22\}:}\varphi$, ${\scriptstyle \{00\}:}p$ for every $p$ such that $V(p) = 00$, ${\scriptstyle \{01,11\}:}p$ for every $p$ such that $V(p) = 11$, and ${\scriptstyle \{02,12,22\}:}p$ for every $p$ such that $V(p) = 22$.*
2. *If $T$ is a tableau and $T'$ is the tree obtained from $T$ applying one of the expansion rules in figure 2, then $T'$ is $\varphi$.*
3. *A branch $B$ in a tableau $T$ is called $V$-*closed *if one of the following condition holds: (i) it contains the constant $\perp$; (ii) it contains* signed literals, ${\scriptstyle S_1:}p, \ldots, {\scriptstyle S_n:}p$, *such that $\cap_{i=1}^{n} S_i = \varnothing$; (iii) all the formulas in the branch have been expanded and, for every variable $p$, it contains signed literals, ${\scriptstyle S_1:}p, \ldots, {\scriptstyle S_n:}p$, such that $\cap_{i=1}^{n} S_i = \{V(p)\}$.*
4. *A tableau $T$ is called $V$-*closed *if every branch is $V$-closed.*

Adding literals of the form ${\scriptstyle \{01,11\}:}p$, ${\scriptstyle \{02,12,22\}:}p$ or ${\scriptstyle \{00\}:}p$, depending on the initial tableau, requires that models be evaluated in a particular form; specifically we force models derived from the tableau to be less than $V$. Nevertheless, we know that one model will always be found, $V$ itself, and therefore we include one more condition on closure: a branch closes if $V$. is the only model generated.

**Theorem 7.** *Let $V$ be a total model of $\varphi$. $V$ is a partial equilibrium model of $\Pi$ if and only if there exists a $V$-closed tableau for $\varphi$.*

In the figure 5 we show that, for the previous example, the model $\sigma_9$ is a partial equilibrium model; observe that the leftmost branch closes because $V$ is the only model generated, while all other branches close due to inconsistencies provoked by the three signed literals added to the initial tableau. In the second tableau in the same figure we check that the model $\sigma_{12}$ is not a partial equilibrium model.

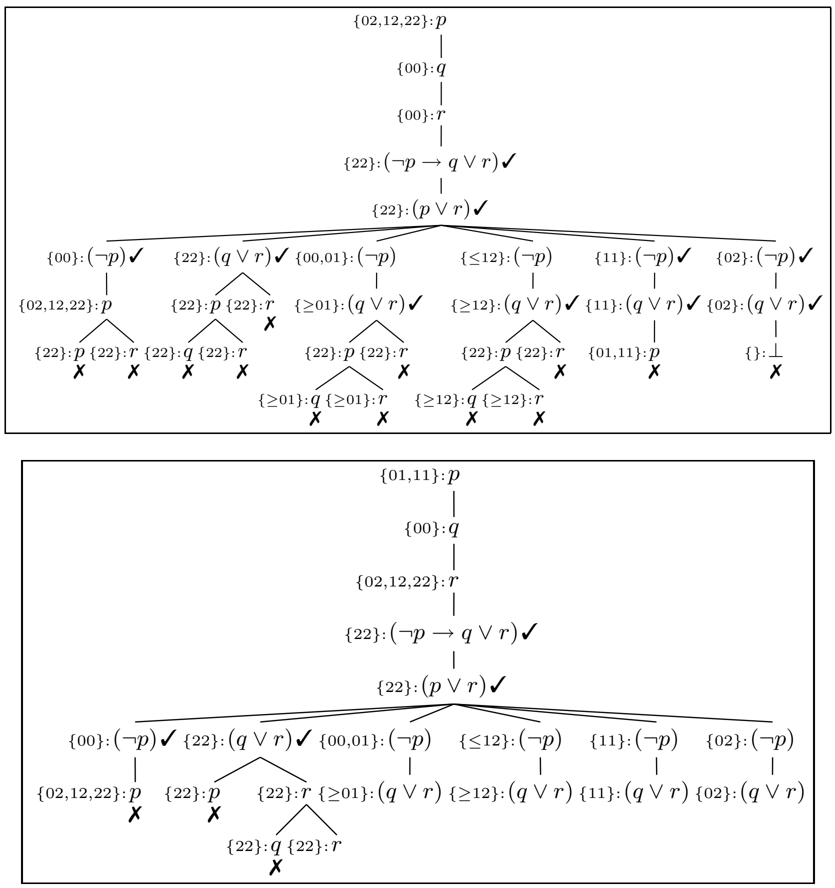

{02,12,22}: $p$
{00}: $q$
{00}: $r$
{22}: $(\neg p \to q \lor r)$ ✓
{22}: $(p \lor r)$ ✓
{00}: $(\neg p)$ ✓
{22}: $(q \lor r)$ ✓
{00,01}: $(\neg p)$
{≤12}: $(\neg p)$
{11}: $(\neg p)$ ✓
{02}: $(\neg p)$ ✓
{02,12,22}: $p$
{22}: $p$ {22}: $r$
{≥01}: $(q \lor r)$ ✓
{≥12}: $(q \lor r)$ ✓
{11}: $(q \lor r)$ ✓
{02}: $(q \lor r)$ ✓
{22}: $p$ {22}: $r$ {22}: $q$ {22}: $r$
{22}: $p$ {22}: $r$
{22}: $p$ {22}: $r$
{01,11}: $p$
{}: $\bot$
{≥01}: $q$ {≥01}: $r$
{≥12}: $q$ {≥12}: $r$
{01,11}: $p$
{00}: $q$
{02,12,22}: $r$
{22}: $(\neg p \to q \lor r)$ ✓
{22}: $(p \lor r)$ ✓
{00}: $(\neg p)$ ✓
{22}: $(q \lor r)$ ✓
{00,01}: $(\neg p)$
{≤12}: $(\neg p)$
{11}: $(\neg p)$
{02}: $(\neg p)$
{02,12,22}: $p$
{22}: $p$
{22}: $r$
{≥01}: $(q \lor r)$
{≥12}: $(q \lor r)$
{11}: $(q \lor r)$
{02}: $(q \lor r)$
{22}: $q$ {22}: $r$

**Fig. 5.**

## 5 A splitting theorem for PEL

The previous tableau calculus offers a general method for satisfiability testing in $HT^2$ and PEL, given any arbitrary theory. When we restrict the syntax to (some class of) logic programs, we usually expect, however, that simpler computation methods can be applied. Consider for instance the case of disjunctive logic programs. As shown in [2], PEL also coincides with p-stable models for this syntactic class. Maintaining the same minimisation criterion, we may easily get that a disjunctive program yields several well-founded models (even no well-founded model at all), and the typical incremental algorithm for computing WFS for normal programs is not applicable. However, it is still possible to apply a form of incremental reasoning if we can divide or "split" the program into blocks without cyclic dependences among them. As an example, consider the simple program $\Pi_0 = \{p \vee q\}$ which yields two p-stable models (also well-founded), making $p$ true and $q$ false in one case, and vice versa. Now, assume we have the enlarged program $\Pi_1 = \Pi_0 \cup \{\neg r \wedge p \to r, q \wedge \neg p \to s, \neg s \to s\}$. It seems natural to use this second set of formulas to compute atoms $r$ and $s$, once $p$ and $q$ are still fixed by the rule in $\Pi_0$. This technique is called "splitting" and was first introduced in [6] for the case of stable models. We now establish a similar result for PEL in the more general syntactic case where theories are sets of implications.

Given a pair $\mathbf{T} = (T, T')$ and a set of atoms $U$, we denote $\mathbf{T}|_U = (T \cap U, T' \cap U)$. We apply a similar notation for theories too. If $\Pi$ is some theory in language $\mathfrak{L}(V)$, and $U \subseteq V$, then we write $\Pi|_U$ to stand for set of formulas $\Pi \cap \mathfrak{L}(U)$. We respectively call *bottom* and *top* to the subtheories $\Pi|_U$ and $\Pi \backslash \Pi|_U$.

**Definition 8 (Splitting set).** *Given a set of implications $\Pi$ on signature $V$, a subset $U \subseteq V$ is called a* splitting set *for $\Pi$ if for all $(\varphi \to \psi) \in \Pi \backslash \Pi|_U$, $\psi \in \mathfrak{L}(V \backslash U)$.* □

**Theorem 8 (Splitting theorem).** *Let $\Pi$ be a set of implications, $U$ a splitting set for $\Pi$ and $\mathbf{T}$ a pair $(T, T')$ of sets of atoms $T \subseteq T'$. Then $\mathbf{T} \models\!\approx \Pi$ iff both* (i) $\mathbf{T}|_U \models\!\approx \Pi|_U$ *and* (ii) $\mathbf{T} \models\!\approx \Pi'$*, being $\Pi' := (\Pi \backslash \Pi|_U)$*

$$\cup\ (T \cap U) \quad (3)$$

$$\cup\ \{\neg p \mid p \in U \backslash T'\} \quad (4)$$

$$\cup\ \{p \leftrightarrow \mathbf{u} \mid p \in (T' \backslash T) \cap U\} \quad (5)$$

*Proof.* "$\Rightarrow$" (i). Assume $\mathbf{T} \models\!\approx \Pi$ but $\mathbf{T}|_U \not\models\!\approx \Pi|_U$. As $\langle \mathbf{T}, \mathbf{T} \rangle \models \Pi$, in particular, $\langle \mathbf{T}, \mathbf{T} \rangle \models \Pi|_U$, but then, clearly $\langle \mathbf{T}|_U, \mathbf{T}|_U \rangle \models \Pi|_U$. Thus, there must exist some $\mathbf{H} = (H, H')$, $\mathbf{H} < \mathbf{T}|_U$ such that $\langle \mathbf{H}, \mathbf{T}|_U \rangle \models \Pi|_U$. Since $\Pi|_U$ exclusively refers to atoms in $U$, this means that any interpretation extending $H, H', T \cap U$ and $T' \cap U$ with atoms not in $U$ will still be a model of that program. In particular, it is easy to see that for $\langle \mathbf{H}_2, \mathbf{T} \rangle$ with $\mathbf{H}_2 = \langle H \cup (T \backslash U), H' \cup (T' \backslash U) \rangle$: (1) it is a well constructed interpretation; (2) it satisfies $\Pi|_U$; (3) it is strictly lower than $\langle \mathbf{T}, \mathbf{T} \rangle$; and (4) for any world $w$ and any formula $\alpha \in \mathfrak{L}(V \backslash U)$, $\langle \mathbf{T}, \mathbf{T} \rangle, w \models \alpha$ iff $\langle \mathbf{H}_2, \mathbf{T} \rangle, w \models \alpha$. As $\mathbf{T} \models\!\approx \Pi$, (2) and (3) mean there must exist some implication $\varphi \to \psi \in \Pi \backslash \Pi|_U$ such that $\langle \mathbf{H}_2, \mathbf{T} \rangle \not\models \varphi \to \psi$. This means $\langle \mathbf{H}_2, \mathbf{T} \rangle, w \models \varphi$ and $\langle \mathbf{H}_2, \mathbf{T} \rangle, w \not\models \psi$ for some world $w$, but as $\langle \mathbf{T}, \mathbf{T} \rangle \models \varphi \to \psi$ this reduces the possibilities to $w \in \{h, h'\}$. Assume $w = h$; then $\langle \mathbf{H}_2, \mathbf{T} \rangle, h \models \varphi$ implying, by the hereditary property, that

$\langle \mathbf{H}_2, \mathbf{T}\rangle, t \models \varphi$ which implies $\langle \mathbf{T}, \mathbf{T}\rangle, h \models \varphi$. This, together with $\langle \mathbf{T}, \mathbf{T}\rangle \models \varphi \rightarrow \psi$ entails $\langle \mathbf{T}, \mathbf{T}\rangle, h \models \psi$, but as $\psi$ does not refer to atoms in $U$, by (4) we obtain $\langle \mathbf{H}_2, \mathbf{T}\rangle, h \models \psi$ reaching a contradiction. The proof for $w = h'$ is completely analogous, replacing $h$ and $t$ by their primed versions in the previous discussion.
"$\Rightarrow$" (ii). Trivially $\langle \mathbf{T}, \mathbf{T}\rangle \models \Pi'$. Assume $\langle \mathbf{H}, \mathbf{T}\rangle \models \Pi'$ with $\mathbf{H} < \mathbf{T}$. It is easy to see that $\Pi'$ fixes the interpretation of atoms in $U$, and so, $\mathbf{H}|_U = \mathbf{T}|_U$ which implies $\langle \mathbf{H}, \mathbf{T}\rangle \models \Pi|_U$. On the other hand, as $\langle \mathbf{H}, \mathbf{T}\rangle$ is model of $\Pi'$, we also have $\langle \mathbf{H}, \mathbf{T}\rangle \models \Pi \backslash \Pi|_U$, so that we get $\langle \mathbf{H}, \mathbf{T}\rangle \models \Pi$ contradicting $\mathbf{T} \approx \Pi$.
"$\Leftarrow$". From (i) and (ii) it is clear that $\langle \mathbf{T}, \mathbf{T}\rangle \models \Pi$. Assume $\langle \mathbf{H}, \mathbf{T}\rangle \models \Pi$ with $\mathbf{H} < \mathbf{T}$. As $\Pi|_U$ only refers to atoms in $U$, $\langle \mathbf{H}|_U, \mathbf{T}|_U\rangle \models \Pi|_U$, but this, together with (i), leaves $\mathbf{H}|_U = \mathbf{T}|_U$ as the only possibility. Now, as $\langle \mathbf{T}, \mathbf{T}\rangle \models (3) \cup (4) \cup (5)$ and these formulas exclusively refer to signature $U$, we obtain $\langle \mathbf{H}, \mathbf{T}\rangle \models (3) \cup (4) \cup (5)$. Finally, as $\langle \mathbf{H}, \mathbf{T}\rangle \models \Pi$ we also have $\langle \mathbf{H}, \mathbf{T}\rangle \models \Pi \backslash \Pi|_U$ and so $\langle \mathbf{H}, \mathbf{T}\rangle \models \Pi'$ contradicting $\mathbf{T} \approx \Pi'$. □

The previous theorem is completed with the following result. Let us denote by $\Pi[\varphi/p]$ the replacement in theory $\Pi$ of any occurrence of atom $p$ by the formula $\varphi$.

**Theorem 9 (Replacement theorem).** *For any theory $\Pi$ and any model $\mathcal{M}$:*
*(i)* $\mathcal{M} \models \Pi \cup \{p\}$ *iff* $\mathcal{M} \models \Pi[\top/p] \cup \{p\}$
*(ii)* $\mathcal{M} \models \Pi \cup \{\neg p\}$ *iff* $\mathcal{M} \models \Pi[\bot/p] \cup \{\neg p\}$
*(iii)* $\mathcal{M} \models \Pi \cup \{p \leftrightarrow \mathbf{u}\}$ *iff* $\mathcal{M} \models \Pi[\mathbf{u}/p] \cup \{p \leftrightarrow \mathbf{u}\}$

*Proof.* For (i) the satisfaction of $p$ at $h$ means that $p$ will hold at any world and so it can be replaced by $\top$. For (ii), the satisfaction of $\neg p$ at $h$ means $p$ will be false at $t'$ and so false at any world, so it can be replaced by $\bot$. Finally, it is easy to see that $\mathcal{M} \models p \leftrightarrow \mathbf{u}$ iff $p$ is true at $h', t'$ but false at $h, t$, which coincides with the valuation of $\mathbf{u}$. □

Returning to the example program $\Pi_1$, $U = \{p, q\}$ is a splitting set dividing $\Pi_1$ into the bottom $\Pi_0$ and the top $\Pi_1 \backslash \Pi_0$. As we saw, $\Pi_0$ has two p-equilibrium models: $\mathbf{T}_1 = (\{p\}, \{p\})$ and $\mathbf{T}_2 = (\{q\}, \{q\})$. Now, fixing $\mathbf{T}_1$, we consider the theory $\Pi' = \Pi_1 \backslash \Pi_0 \cup \{p\} \cup \{\neg q\}$ which, by the replacement theorem, is equivalent to $\{\neg r \wedge \top \rightarrow r, \bot \wedge \neg \top \rightarrow s, \neg s \rightarrow s, p, \neg q\}$. After some trivial simplifications, this amounts to $\{\neg r \rightarrow r, \neg s \rightarrow s, p, \neg q\}$ whose unique p-equilibrium model is defined by $\mathbf{T}_3 = (\{p\}, \{p, r, s\})$. Following similar steps, when fixing $\mathbf{T}_2$ we finally get the program $\{s, \neg s \rightarrow s, q, \neg p\}$ with the only p-equilibrium model $\mathbf{T}_4 = (\{q, s\}, \{q, s\})$.

## 6 Concluding remarks

Partial equilibrium logic (PEL) provides a foundation and generalisation of the p-stable semantics of logic programs and hence is arguably also a suitable framework for studying the well-founded semantics of programs. In this paper we have extended previous results on PEL by further examining its underlying logics $HT^2$ and $HT^*$, and presenting tableaux proof systems for $HT^2$, $HT^*$ and for PEL itself. As a contribution to the computation of PEL in the case of disjunctive and nested logic programs, we have shown how to apply the splitting method of [6,3]. Further optimisation of these computational techniques is a topic for future work.

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