Probabilistic Autoepistemic Equilibrium Logic*

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Abstract. In this short note, we consider the definition of an *Probabilistic Epistemic Logic* (PE) and its non-monotonic extension, we call *Probabilistic Autoepistemic Equilibrium Logic* (PAEE). PE introduces a probabilistic modality that allows expressing lower bounds on conditional probability constructs. Regular (non-probabilistic) modal epistemic operators $\mathbf{K}$ and $\mathbf{M}$ can be defined as derived constructs in PE so that, in fact, for that modal epistemic fragment, PE collapses into modal logic KD45. The non-monotonic extension of PE follows the same steps as *Equilibrium Logic* [7], the main logical characterisation of Answer Set Programming [1], ASP. Equilibrium logic consists in a selection among the models of a theory under the intermediate logic of *Here-and-There* (HT) [5]. Similarly, we define the combination of PE with HT, we call PEHT, and then, define a model selection criterion that gives rise to the non-monotonic formalism of Autoepistemic PE. We end up showing that, if we consider again the modal epistemic fragment of the syntax, PAEE collapses to *Epistemic Specifications* [4], a well-known epistemic extension of ASP and we use a previous result to illustrate how the further addition of the excluded middle axiom, eventually produces *Autoepistemic Logic* [6] as a particular case.

1 Syntax and Semantics of PE

The syntax starts from some (countable) set of atoms At we call the propositional *signature*. Formulas φ are defined following the syntax:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \mathbf{P}(\varphi \,|\, \varphi) \geq x$$

where $p \in At$ and x is any constant real number $x \in [0,1]$. A *theory* Γ is a set of formulas. Intuitively, the reading of the modal operator $\mathbf{P}(\varphi \,|\, \psi) \geq x$ is:

"the probability of φ conditioned to ψ is at least x".

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We will also use the following derived propositional operators $\neg\varphi \stackrel{\text{def}}{=} (\varphi \rightarrow \bot)$, $\top \stackrel{\text{def}}{=} \neg\bot$ and $\varphi \leftrightarrow \psi \stackrel{\text{def}}{=} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ plus a modal derived operator $\mathbf{P}(\varphi \,|\, \psi) \succ x$ that is defined as:

$$\mathbf{P}(\varphi \,|\, \psi) \succ x \stackrel{\text{def}}{=} \neg(\, \mathbf{P}(\neg\varphi \,|\, \psi) \succeq 1 - x \,)$$

and whose intuitive reading is:

"the probability of φ conditioned to ψ is greater than x".

We will sometimes use the construct $\mathbf{P}(\varphi)$ as an abbreviation of $\mathbf{P}(\varphi \,|\, \top)$ regardless of the comparison symbol we use $\succeq$ or $\succ$. In fact, new derived operators using other comparison symbols will be introduced later on. Intuitively, $\mathbf{P}(\varphi \,|\, \psi) \succeq x$ is a modal necessity operator, whereas $\mathbf{P}(\varphi \,|\, \psi) \succ x$ is its dual possibility operator.

As always, a *propositional interpretation* I is a set of atoms from the signature, $I \subseteq At$. Each possible interpretation can be seen as a different state of affairs of the real world. To represent the agent's beliefs we will use a probability distribution over these states of affairs. A *(probabilistic) belief view* π is a probability distribution over interpretations $\pi : 2^{At} \rightarrow [0,1]$ so that $\sum_{I \subseteq At} \pi(I) = 1$. The set of *worlds for* π is defined as $W_\pi \stackrel{\text{def}}{=} \{ I \subseteq At \mid \pi(I) > 0 \}$, so it collects all interpretations that are assigned a strictly positive probability by belief view π, that is, those that effectively are among the agent's beliefs, with some non-zero probability. Note that W_π cannot be empty[4], since the sum of all probabilities for interpretations must be 1.

A *belief interpretation* is a pair (π, I) where π is a belief view and $I \subseteq At$ is an interpretation that accounts for the *real world*. Note that π represents beliefs and not knowledge: as a result, it may be the case that $I \notin W_\pi$, that is, the agent may believe that the real world I has a probability $\pi(I) = 0$.

Definition 1 (Satisfaction). *We define the satisfaction of formula φ by a belief interpretation (π, I), written $(\pi, I) \models \varphi$, recursively as follows:*

1. $(\pi, I) \not\models \bot$
2. $(\pi, I) \models p$ *if* $p \in I$
3. $(\pi, I) \models \varphi \wedge \psi$ *if* $(\pi, I) \models \varphi$ *and* $(\pi, I) \models \psi$
4. $(\pi, I) \models \varphi \vee \psi$ *if* $(\pi, I) \models \varphi$ *or* $(\pi, I) \models \psi$
5. $(\pi, I) \models \varphi \rightarrow \psi$ *if* $\pi, I \not\models \varphi$ *or* $(\pi, I) \models \psi$
6. $(\pi, I) \models \mathbf{P}(\varphi \,|\, \psi) \succeq x$ *if* $\pi(\psi) = 0$ *or* $\pi(\varphi \wedge \psi)/\pi(\psi) \geqslant x$

where the application of $\pi(\phi)$ on any formula ϕ we used in the last item simply stands for:

$$\pi(\phi) \stackrel{\text{def}}{=} \sum \{ \pi(J) \mid J \subseteq 2^{\mathrm{At}}, (\pi, J) \models \phi \}$$

□

According to this definition, it is easy to see that $\pi(\bot) = 0$, $\pi(\neg\varphi) = 1 - \pi(\varphi)$ and $\pi(\top) = 1$. In this context, implication $\varphi \rightarrow \psi$ is classical and amounts to $\neg\varphi \vee \psi$. Similarly, the satisfaction of negation $(\pi, I) \models \neg\varphi$ amounts to $(\pi, I) \not\models \varphi$.

[4] This holds even for $At = \varnothing$, where the only possible interpretation would be $\varnothing$ and the only possible belief view assigns $\pi(\varnothing) = 1$, so $W_\pi = \{\varnothing\}$.

Proposition 1. *$(\pi, I) \models \mathbf{P}(\varphi \,|\, \psi) \succ x$ iff both $\pi(\psi) \neq 0$ and $\pi(\varphi \wedge \psi)/\pi(\psi) > x$.*

Proof. We start observing:

$$\begin{aligned}(\pi, I) \models \mathbf{P}(\varphi \,|\, \psi) \succ x &\Leftrightarrow (\pi, I) \models \neg\mathbf{P}(\neg\varphi \,|\, \psi) \succeq 1 - x\\ &\Leftrightarrow (\pi, I) \not\models \mathbf{P}(\neg\varphi \,|\, \psi) \succeq 1 - x\\ &\Leftrightarrow \pi(\psi) \neq 0 \text{ and } \pi(\neg\varphi \wedge \psi)/\pi(\psi) < 1 - x\\ &\Leftrightarrow \pi(\psi) \neq 0 \text{ and } \pi(\neg\varphi \wedge \psi) < (1 - x) \cdot \pi(\psi)\end{aligned}$$

We will prove that, when $\pi(\psi) \neq 0$, the last conjunct is equivalent to $\pi(\varphi \wedge \psi) > x \cdot \pi(\psi)$. To this aim, note that $\pi(\psi) = \pi(\varphi \wedge \psi) + \pi(\neg\varphi \wedge \psi)$ and so:

$$\begin{aligned}&\pi(\neg\varphi \wedge \psi) < (1 - x) \cdot \pi(\psi)\\ \Leftrightarrow\ &\pi(\neg\varphi \wedge \psi) < (1 - x) \cdot \pi(\varphi \wedge \psi) + (1 - x) \cdot \pi(\neg\varphi \wedge \psi)\\ \Leftrightarrow\ &\pi(\neg\varphi \wedge \psi) - (1 - x) \cdot \pi(\neg\varphi \wedge \psi) < \pi(\varphi \wedge \psi) - x \cdot \pi(\varphi \wedge \psi)\\ \Leftrightarrow\ &x \cdot \pi(\neg\varphi \wedge \psi) < \pi(\varphi \wedge \psi) - x \cdot \pi(\varphi \wedge \psi)\\ \Leftrightarrow\ &x \cdot (\pi(\neg\varphi \wedge \psi) + \pi(\varphi \wedge \psi)) < \pi(\varphi \wedge \psi)\\ \Leftrightarrow\ &x \cdot \pi(\psi) < \pi(\varphi \wedge \psi)\end{aligned}$$

□

As a result of Proposition 1, given that $\pi(\top) = 1$, it is easy to see that:

$$\mathbf{P}(\varphi) \succeq x \;\Leftrightarrow\; \mathbf{P}(\varphi \,|\, \top) \succeq x \;\Leftrightarrow\; \pi(\varphi) \geqslant x$$
$$\mathbf{P}(\varphi) \succ x \;\Leftrightarrow\; \mathbf{P}(\varphi \,|\, \top) \succ x \;\Leftrightarrow\; \pi(\varphi) > x$$

A belief interpretation (π, I) is a *belief model* of a theory Γ if $(\pi, J) \models \varphi$ for all $\varphi \in \Gamma$ and $J \in W_\pi \cup \{I\}$. We say that a belief view π is an *epistemic model* of a theory Γ, abbreviated as $\pi \models \Gamma$, when $(\pi, J) \models \varphi$ for all $\varphi \in \Gamma$ and all $J \in W_\pi$. A formula φ is a *tautology* if $(\pi, I) \models \varphi$ for any belief interpretation (π, I). We call *Probabilistic Epistemic Logic* (PE) to the logic induced by all tautologies.

The following are some interesting derived operators and their induced semantics:

$$\begin{aligned}
\mathbf{K}\varphi &\stackrel{\text{def}}{=} \mathbf{P}(\varphi) \succeq 1 && \Leftrightarrow\ (\pi, I) \models \varphi \text{ for all } I \in W_\pi\\
\mathbf{M}\varphi &\stackrel{\text{def}}{=} \mathbf{P}(\varphi) \succ 0 && \Leftrightarrow\ (\pi, I) \models \varphi \text{ for some } I \in W_\pi\\
\mathbf{P}(\varphi \,|\, \psi) \preceq x &\stackrel{\text{def}}{=} \neg(\mathbf{P}(\varphi \,|\, \psi) \succ x) && \Leftrightarrow\ \pi(\psi) = 0 \text{ or } \pi(\varphi \wedge \psi)/\pi(\psi) \leqslant x\\
\mathbf{P}(\varphi \,|\, \psi) \prec x &\stackrel{\text{def}}{=} \neg(\mathbf{P}(\varphi \,|\, \psi) \succeq x) && \Leftrightarrow\ \pi(\psi) > 0 \text{ and } \pi(\varphi \wedge \psi)/\pi(\psi) < x\\
\mathbf{P}(\varphi \,|\, \psi) \hat{=} x &\stackrel{\text{def}}{=} \mathbf{P}(\varphi \,|\, \psi) \succeq x &&\\
&\qquad \wedge \mathbf{P}(\varphi \,|\, \psi) \preceq x && \Leftrightarrow\ \pi(\psi) = 0 \text{ or } \pi(\varphi \wedge \psi)/\pi(\psi) = x\\
\mathbf{P}(\varphi \,|\, \psi) = 1 &\stackrel{\text{def}}{=} \mathbf{P}(\varphi \,|\, \psi) \succeq 1 && \Leftrightarrow\ \mathbf{K}(\psi \rightarrow \phi)\\
\mathbf{P}(\varphi \,|\, \psi) = x &\stackrel{\text{def}}{=} \mathbf{M}\psi \wedge \mathbf{P}(\varphi \,|\, \psi) \hat{=} x && \Leftrightarrow\ \pi(\psi) > 0 \text{ and } \pi(\varphi \wedge \psi)/\pi(\psi) = x\\
&&& \quad \text{for } x < 1
\end{aligned}$$

Notice that $\mathbf{P}(\varphi \,|\, \psi) \hat{=} x$ is a weak assertion in the sense that it is trivially true when $\pi(\psi) = 0$ (that is, there are no worlds satisfying ψ). The stronger version $\mathbf{P}(\varphi \,|\, \psi) = x$ depends on the value chosen for x. If $x = 1$ this amounts to checking $\mathbf{P}(\varphi \,|\, \psi) \succeq 1$ because a probability cannot have a value larger than 1. Note that this formula is equivalent to $\mathbf{K}(\psi \rightarrow \varphi)$, that is, we just check that all worlds satisfying ψ also satisfy φ. When $\pi(\psi) = 0$, there are no worlds in which ψ and the probability of the conditional is trivially $x = 1$. For this reason, when $x < 1$, we must have $\pi(\psi) > 0$ because, as we just said, $\pi(\psi) = 0$ would make the conditional trivially true and require probability $x = 1$. The formula $\mathbf{M}\,\psi$ is used to force $\pi(\psi) > 0$, that together with $\mathbf{P}(\varphi \,|\, \psi) \hat{=} x$, produces the expected result. We can also observe that:

$$\begin{array}{rcl} \mathbf{P}(\varphi \,|\, \psi) \succeq 1 & \Leftrightarrow & \mathbf{K}(\psi \rightarrow \varphi) \\ \mathbf{P}(\varphi \,|\, \psi) \succ 0 & \Leftrightarrow & \mathbf{M}(\varphi \wedge \psi) \end{array}$$

We say that a formula is *epistemic* when all its modal opearors are of the form $\mathbf{K}$ or $\mathbf{M}$. By a simple inspection of the derived semantics for $\mathbf{K}$ and $\mathbf{M}$, the following result can be easily checked:

Theorem 1. *Let φ be an epistemic formula. Then $(\pi, I) \models \varphi$ iff $W_\pi, I \models \varphi$ in modal logic KD45.* □

2 Probabilistic Autoepistemic Equilibrium Logic

We define now the combination of PE with the intermediate logic of HT. In the latter, interpretations have the form of pairs $\langle H, T \rangle$ of sets of atoms where H (called the "here" world) is a subset of T (the "there" world). We define an PEHT-*interpretation* as a triple $\langle \pi, H, T \rangle$ where $H \subseteq T \subseteq At$ and π is a belief view. When $H = T$ we say that the interpretation is *total* and we just write it as a pair $\langle \pi, T \rangle$.

Definition 2 (PEHT-satisfaction). *A PEHT-interpretation* satisfies *a formula φ, written $\langle \pi, H, T \rangle \models \varphi$, if the following recursive conditions hold:*

- $\langle \pi, H, T \rangle \not\models \bot$
- $\langle \pi, H, T \rangle \models p$ *iff* $p \in H$
- $\langle \pi, H, T \rangle \models \varphi \wedge \psi$ *iff* $\langle \pi, H, T \rangle \models \varphi$ *and* $\langle \pi, H, T \rangle \models \psi$
- $\langle \pi, H, T \rangle \models \varphi \vee \psi$ *iff* $\langle \pi, H, T \rangle \models \varphi$ *or* $\langle \pi, H, T \rangle \models \psi$
- $\langle \pi, H, T \rangle \models \varphi \rightarrow \psi$ *iff both:*
  (i) $\langle \pi, T \rangle \models \varphi \rightarrow \psi$ *and*
  (ii) $\langle \pi, H, T \rangle \not\models \varphi$ *or* $\langle \pi, H, T \rangle \models \psi$
- $\langle \pi, H, T \rangle \models \mathbf{P}(\varphi \,|\, \psi) \succeq x$ *if* $\pi(\psi) = 0$ *or* $\pi(\varphi \wedge \psi)/\pi(\psi) \geqslant x$

where the application of $\pi(\phi)$ on any formula ϕ we used in the last item simply stands for:

$$\pi(\phi) \stackrel{\text{def}}{=} \sum \{\pi(J) \mid J \subseteq 2^{\mathrm{At}}, \langle \pi, T \rangle \models \phi\}$$

□

As usual, we say that $\langle \pi, H, T \rangle$ is a *model* of a theory Γ, in symbols $\langle \pi, H, T \rangle \models \Gamma$, iff $\langle \pi, H, T \rangle \models \varphi$ for all $\varphi \in \Gamma$. We define PEHT-tautologies as formulas satisfied by every PEHT-interpretation, as expected. PEHT is the logic induced by all PEHT-tautologies.

Definition 3 (Equilibrium model). *A set of atoms T is a* π-equilibrium model *of a theory Γ if $\langle \pi, T \rangle \models \Gamma$ and there is no $H \subset T$ such that $\langle \pi, H, T \rangle \models \Gamma$.* □

We denote the set of π-equilibrium models of Γ as $EQ[\pi, \Gamma]$.

Definition 4 (Probabilistic world view). *A belief view π is a* probabilistic world view *for a theory Γ if:*

$$W_\pi = EQ[\pi, \Gamma]$$

□

We define the Probabilistic Autoepistemic Equilibrium Logic (PAEE) as the logic induced by probabilistic world views.

Epistemic Specifications were defined by Gelfond in [4] for an extension of logic programs with epistemic literals in the ruel conditions (or bodies). In [2], a straightforward extension for covering the syntax of arbitrary epistemic formulas was provided.

Theorem 2. *Let Γ be an epistemic theory and π some belief view. Then π is a probabilistic world view of Γ iff W_π is a world view of Γ in the sense of epistemic specifications as in [2].*

This relation is one-to-many. We may have several π with the same W_π. This also means, for instance, that when we look at the worlds W_π induced by each probabilistic world view π in PAEE and we restrict the syntax to epistemic specifications, we essentially get Gelfond's world views as originally defined in [4].

Moreover, according to Proposition 1 in [2], the world views[5] of an epistemic theory Γ we get from Autoepistemic Logic [6] just correspond to Gelfond's world views for $\Gamma \cup (EM)$ where (EM) stands for the axiom of excluded middle:

$$p \vee \neg p \tag{EM}$$

for every atom $p \in At$. As a consequence, if we consider PAEEplus the (EM) axiom we obtain a *probabilistic proper extension* of Autoepistemic Logic.

3 Conclusions

We have presented an expressive non-monotonic formalism, PAEE, whose monotonic basis PEHT constitutes a combination of the logic of Here-and-There plus the well-known modal logic KD45, but further generalised for dealing with probabilities. Similarly, the non-monotonic formalism, PAEE, constitutes a probabilisitc generalisation of Gelfond's epistemic specifications and, when the excluded-middle axiom is added, of Moore's Autoepistemic Logic.

[5] Actually called *theory expansions* in the original terminology [6].

For future work, we plan to investigate the representation of probabilistic independence among atoms or formulas and the incorporation of the principle of indifference (without further information, all states of affairs have equal probability). Also, we plan to investigate the formal relation to the probabilisitc logic programming formalism of ProbLog [8] and to other modal approaches for probabilities such as [3].

References

1. Brewka, G., Eiter, T., Truszczyński, M.: Answer set programming at a glance. Communications of the ACM 54(12), 92–103 (2011)
2. Cabalar, P., Fandinno, J., Fariñas del Cerro, L.: Autoepistemic answer set programming. Artificial Intelligence 289, 103382 (2020), https://doi.org/10.1016/j.artint.2020.103382
3. Fagin, R., Halpern, J.Y., Megiddo, N.: A logic for reasoning about probabilities. Inf. Comput. 87(1/2), 78–128 (1990), https://doi.org/10.1016/0890-5401(90)90060-U
4. Gelfond, M.: Logic programming and reasoning with incomplete information. Annals of Mathematcis and Artificial Intelligence 12(1-2), 89–116 (1994), https://doi.org/10.1007/BF01530762
5. Heyting, A.: Die formalen Regeln der intuitionistischen Logik. Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse pp. 42–56 (1930)
6. Moore, R.C.: Semantical considerations on nonmonotonic logic. Artif. Intell. 25(1), 75–94 (1985), https://doi.org/10.1016/0004-3702(85)90042-6
7. Pearce, D.: A new logical characterisation of stable models and answer sets. In: NMELP. Lecture Notes in Computer Science, vol. 1216, pp. 57–70. Springer (1997)
8. Raedt, L.D., Kimmig, A., Toivonen, H.: Problog: A probabilistic prolog and its application in link discovery. In: Veloso, M.M. (ed.) IJCAI 2007, Proceedings of the 20th International Joint Conference on Artificial Intelligence, Hyderabad, India, January 6-12, 2007. pp. 2462–2467 (2007), http://ijcai.org/Proceedings/07/Papers/396.pdf