EXTENDED MEREOTOPOLOGY BASED ON SEQUENT ALGEBRAS: Mereotopological representation of Scott and Tarski consequence relations

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INTRODUCTION

In this talk we will give a spatial representation of Skott and Tarski consequence relations in **mereotopology**. It is an extension of mereology with some binary relations of topological nature. Mereotopology is considered also as a kind of **pointless geometry**, called also **region-based theory of space**. Its intended models are Boolean algebras of regular closed (or open) sets in topological spaces, called **regions** with an additional binary relation C between regions called **contact**. The intuitive meaning of aCb is "a and b share a common point". The algebraic form of the theory is a Boolean algebra B with an additional relation C (called **contact algebra**) – $B = (B, 0, 1, \leq, +, ., *, C)$.

In the generalization which we will consider, contact C is replaced with the Scott consequence relation $A \vdash B$, where $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ are finite sets of regions. The intuitive spatial meaning of $\vdash$ is

$\{a_1 \cap \ldots \cap a_n\} \subseteq \{b_1 \cup \ldots \cup b_m\}$

Now the standard (binary) contact can be defined:

aCb iff $a, b \nvdash \varnothing$

We can define also an n-ary contact for each n:

$C_n(a_1, \ldots, a_n)$ iff $a_1, \ldots, a_n \nvdash \varnothing$

THE AIM OF THIS TALK is to give axiomatic definition of Boolean algebras with Scott consequence relation, to be called later on S-algebras, and to prove for S-algebras the intended topological representation theorem.

PLAN OF THE TALK

1. Some references and short history.
2. Contact algebras
3. Scott and Tarski consequence relations
4. Sequential algebras (S-algebras)
5. Representation theory for S-algebras
6. S-algebras with additional axioms
7. Propositional logics based on S-algebras

1. REFERENCES and SHORT HISTORY

I. Early history of region-based theory of space: de Laguna (1922), Whitehead (Proces and Reality 1929), Tarski (1927). Relations to MEREOLOGY

II. Some papers for contact algebras and region-based theory of space

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Papers on logics based on contact algebras.

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Mainly, we will use notions and results from Dimov and Vakarelov [8] for the representation theory of contact algebras and from Dimov and Vakarelov [16] - for Scott and Tarski consequence relations and [18] - for spatial logics.

2. CONTACT ALGEBRAS

Definition. By a **Contact Algebra** (CA) we will mean any system $\underline{B} = (B, C) = (B, 0, 1, ., +, *, C)$ where $(B, 0, 1, ., +, *)$ is a non-degenerate Boolean algebra with a complement denoted by "$*$" and C – a binary relation in B, called *contact* and satisfying the following axioms:

$(C1)$ $\quad xCy \rightarrow x, y \neq 0$,

$(C2)$ $\quad xCy \rightarrow yCx$,

$(C3)$ $\quad xC(y + z) \leftrightarrow xCy$ or xCz,

$(C4)$ $\quad x.y \neq 0 \rightarrow xCy$.

We say that $\underline{B}$ is *complete* if the underlying Boolean algebra B is complete.

Contact algebras may satisfy some additional axioms:

(Con) $a \neq 0, 1 \rightarrow aCa^*$ – the axiom of **connectedness**

(Ext) $(\forall a \neq 1)(\exists b \neq 0)(a\overline{C}b)$ – the axiom of **extensionality**

This axiom is equivalent to the following condition assumed by Whitehead:

$a = b$ iff $(\forall c)(aCc \leftrightarrow bCc)$

(Nor) $a\overline{C}b \rightarrow (\exists c)(a\overline{C}c$ and $b\overline{C}c^*)$ – the axiom of **normality**

Examples of CA-s

1. Topological example: the CA of regular closed sets.

Let (X, τ) be an arbitrary topological space. A subset a of X is *regular closed* if $a = Cl(Int(a))$. The set of all regular closed subsets of (X, τ) will be denoted by $RC(X)$. It is a well-known fact that regular closed sets with the operations

$a + b = a \cup b$, $a.b = Cl(Int(a \cap b))$,

$a^* = Cl(X \setminus a)$, $0 = \varnothing$ and $1 = X$

form a Boolean algebra. If we define the contact by $a\,C_X\,b$ iff $a \cap b \neq \varnothing$

Then we have:

Lemma. $RC(X)$ with the above contact is a contact algebra. If X is a connected space then $RC(X)$ satisfies (Con). If X is normal space then $(RC(X)$ satisfies the axiom (Nor). The CA of this example is said to be **standard contact algebra of regular closed sets**.

2. Non-topological example, related to Kripke semantics of modal logic. Let (X, R) be a reflexive and symmetric modal frame and let $B(X)$ be the Boolean algebra of all subsets of X. Define a contact $C_R b$ between two subsets $a, b \in B(X)$ by $aC_R b$ iff $(\exists x \in a)(\exists y \in b)(xRy)$. Then we have:

Lemma.(Duentsch, Vakarelov [6]) $B(X)$ equipped with the contact C_R is a contact algebra. If R is (X, R) is a connected graph, then $B(X)$ satisfies the axiom (Con). If R is transitive, then $B(X)$ satisfies the axiom (Nor).

Examples of contact algebras based on Kripke frames are called **discrete CA**. They are related to a version of discrete region-based theory of space, based on the so called **adjacency spaces**, proposed by Galton. Adjacency spaces are just the above Kripke sstructures (X, R) where R is treated intuitively as the **adjacency relation**.

A topological space X is called **semiregular** if it has a closed base of regular closed sets.

Representation theorem for contact algebras (Dimov, Vakarelov [8]). For each contact algebra $\underline{B} = (B, C)$ there exists a compact semiregular T_0 space X and a dense embedding h of $\underline{B}$ into the contact algebra $RC(X)$. If $\underline{B}$ satisfies (Con) then the space X is connected and $RC(X)$ satisfies (Con). If B is a complete algebra then h is an isomorphism onto $RC(X)$.

Similar representation theorems for contact algebras satisfying some of the other additional axioms are also true (see [5,6,7,8]).

Idea of the proof. The key is how to define abstract points in CA and semiregular topology in the set of points. Abstract points are called **clans** - a construction taken from the theory of proximity spaces.

Definition. A subset $\Gamma \subseteq B$ is called a **clan** if it satisfies the following conditions:

(Clan1) $1 \in \Gamma$, $0 \notin \Gamma$,

(Clan 2) If $a \in \Gamma$ and $a \leq b$ then $b \in \Gamma$,

(Clan 3) If $a, b \in \Gamma$ then aCb,

(Clan 4) If $a + b \in \Gamma$ then $a \in \Gamma$ or $b \in \Gamma$.

Examples of clans:

The set $\Gamma_x = \{a \in RC(X) : x \in a\}$ is a clan (called point-clan). All prime filters in $\underline{B}$ are clans, but there are clans which are not prime filters. Denote the set of clans of $\underline{B}$ by $CLANS(B)$.

Define $h(a) = \{\Gamma \in CLANS(B) : a \in \Gamma\}$.

Take the set $\{h(a) : a \in B\}$ as a base for the topology in the set $X = CLANS(B)$.

Lemma.

(I) (1) aCb iff $(\exists\Gamma \in CLANS(B))(a, b \in \Gamma)$ iff $h(a) \cap h(b) \neq \varnothing$

(2) $h(a + b) = h(a) \cup h(b)$,

(3) $h(a^*) = Cl(Int(h(a)))$,

(4) $h(a)$ is a regular closed set,

(II) The space $X = CLANS(B)$ is semiregular, compact and T_0 and h is a dense embedding into the contact algebra $RC(X)$. If $\underline{B}$ is complete then h is an isomorphism with the complete contact algebra $RC(X)$. If $\underline{B}$ satisfies (Con) then X is a connected topological space and $RC(X)$ also satisfies (CON).

Obviously the Lemma implies the Theorem.

If $\underline{B}$ satisfies the axiom (Ext) then points are maximal clans and the space is compact and T_1 with the property that $RC(X)$ satisfies (Ext).

If the algebra satisfies both (Ext) and (Nor) then the points are special maximal clans, called clusters and the space is compact and T_2 (see [5,8]).

If the algebra is complete then a one-one correspondence between contact algebras (with some of the additional axioms) and the corresponding representation spaces is found up to isomorphism between algebras and homeomorphism between spaces(see [8]).

3. Scott and Tarski sequent systems

Let A, B be finite sets. Then the expression $A \vdash B$ is called **sequent**. We adopt all standard notations and abbreviations from the Sequent calculus.

Definition.(Scott sequent system) The system $S = (W, \vdash)$, where $W \neq \varnothing$ and $\vdash$ is a binary relation between finite subsets of W, is called Scott sequent system (Scott S-system), if the the following axioms are satisfied (A, B are finite subsets of W and $x \in W$) :

(Ref) $x \vdash x$,

(Mono) If $A \vdash B$, then $A, x \vdash B$ and $A \vdash x, B$,

(Cut) If $A, x \vdash B$ and $A \vdash x, B$ then $A \vdash B$.

The relation $\vdash$ is extended for arbitrary subsets of W by the following definition: $A \vdash B$ iff for some finite subsets $A' \subseteq A$ and $B' \subseteq B$ we have $A' \vdash B'$. The axioms (Ref), (Mono) and (Cut) also hold for the extended relation.

Example. Let W be a nonempty set whose elements are sets.

Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$.

Define $A \vdash B$ iff $a_1 \cap \ldots \cap a_n \subseteq b_1 \cup \ldots \cup b_m$. Then the system $(W, \vdash)$ is an S-system.

If we consider the restriction of the relation $A \vdash B$ for the case $B = \{b\}$ (vrite $A \vdash b$) then this restriction of $\vdash$ is called **Tarski consequence relation**. It can be characterized axiomatically as follows:

Definition. (Tarski sequent system) The system $S = (W, \vdash)$, where $W \neq \varnothing$ and $\vdash$ is a binary relation $A \vdash b$ with A a finite subset of W and $b \in W$, is called Tarski sequent system (Tarski S-system), if the the following axioms are satisfied (A is a finite subset of W and $x, y \in W$):

(Ref) $x \vdash x$,

(Mono) If $A \vdash x$, then $A, y \vdash x$,

(Cut) If $A \vdash x$, $A, x \vdash y$, then $A \vdash y$.

SEQUENTIAL ALGEBRAS (S-algebras).

Definition (Scott version).
We call the system $\underline{B} = (B, 0, 1, \leq, +, ., *, \vdash)$ **sequential Boolean algebra (S-algebra)** if $(B, 0, 1, \leq, +, ., *)$ is a non-degenerate Boolean algebra, $(B, \vdash)$ is a Scott S-system and $\vdash$ satisfies the following additional axioms:

(S1) $x \vdash y$ iff $x \leq y$,

(S2) $\varnothing \vdash y$ iff $1 \vdash y$,

(S3) If $A, x \vdash B$ and $A, y \vdash B$, then $A, x + y \vdash B$,

(S4) $A \vdash x_1, \ldots, x_m$ iff $A \vdash x_1 + \cdots + x_m$. (The case $m = 0$ is included: $A \vdash \varnothing$ iff $A \vdash 0$.)

Axiom (S4) reduces the Scott consequence relation to the Tarski consequence relation, which makes possible to give an equivalent and simpler definition, based on Tarski consequence relation.

Definition (Tarski version).
We call the system $\underline{B} = (B, 0, 1, \leq, +, ., *, \vdash)$ **sequential Boolean algebra (S-algebra)** if $(B, 0, 1, \leq, +, ., *)$ is a non-degenerate Boolean algebra, $(B, \vdash)$ is a Tarski S-system and $\vdash$ satisfies the following additional axioms:

(T1) $x \vdash y$ iff $x \leq y$,

(T2) $\varnothing \vdash y$ iff $1 \vdash y$,

(T3) If $A, x \vdash z$ and $A, y \vdash z$, then $A, x + y \vdash z$,

Scott consequence relation is now definable:

$A \vdash x_1, \ldots, x_m$ iff $A \vdash x_1 + \cdots + x_m$.

Examples of S-algebras.

1. Topological S-algebras. Let X be a topological space and $RC(X)$ be the Boolean algebra of regular closed sets of X.

Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$

be two finite subsets of $RC(X)$. Define

$A \vdash B$ iff $a_1 \cap \ldots \cap a_n \subseteq b_1 \cup \ldots \cup b_m$.

Lemma $RC(X)$ equipped with this relation is an S-algebra (based on Scott consequence relation).

The modification of this construction for Tarski consequence relation is obvious:

$A \vdash b$ iff $a_1 \cap \ldots \cap a_n \subseteq b$.

The examples of the above kind are called **standard topological S-algebras**.

2. Non-topological examples (Discrete S-algebras).

Let (X, Y) be a pair (called a **discrete S-space**) with X a non-empty set and Y a set of subsets of X containing all singletons of X. Let $B(X, Y)$ be the Boolean algebra of subsets of X.

We define Tarski-type consequence relation in $B(X, Y)$ as follows $(a_1, \ldots, a_n, b \in B(X, Y))$:

$a_1, \ldots, a_n \vdash_{dicr} b$ iff
$(\forall x_1 \ldots x_n \in X, \Gamma \in Y)(\{x_1 \ldots x_n\} \subseteq \Gamma,$
$x_1 \in a_1, \ldots, x_n \in a_n \to b \cap \Gamma \neq \varnothing)$

Lemma. $B(X, Y)$ equipped with the above Tarski consequence relation is an S-algebra, called the **discrete S-algebra over the discrete S-space (X, Y).**

Topological and discrete examples are in a sense characteristic because every S-algebra is representable in both senses.

5. REPRESENTATION THEORY FOR S-ALGEBRAS

In order to develop a representation theory for S-algebras we need a suitable notion of abstract point. We generalize the notion of clan from the case of contact algebras. From now on we will assume that the consequence relation in $\underline{B}$ is of Tarski form.

S-clans. Let $\underline{B}$ be an S-algebra. A subset $\Gamma \subseteq B$ is called an S-clan if it satisfies the following conditions:

(S-clan 1) $1 \in \Gamma$, $0 \notin \Gamma$,

(S-clan 2) If $a_1, \ldots, a_n \in \Gamma$ and $a_1 \ldots, a_n \vdash b$, then $b \in \Gamma$,

(S-clan 3) If $a + b \in \Gamma$ then $a \in \Gamma$ or $b \in \Gamma$.

S-clans which are maximal with respect to set-inclusion are called maximal S-clans. Every S-clan can be extended into a maximal S-clan (Zorn). We denote by $SCLANS(B)$ and by $MaxSCLANS(B)$ the sets of all S-clans and maximal S-clans of B. By $Ult(\Gamma)$ we denote the set of all ultrafilters contained in the S-clan Γ, $Ult(B)$ denote the set of all ultrafilters in $\underline{B}$.

Examples of S-clans. 1. Let X be a topological space and $x \in X$. Then $\Gamma_x = \{a \in RC(X) : x \in a\}$ is an S-clan called a point S-clan.

2. Every ultrafilter in $\underline{B}$ is an S-clan.

Note that every S-clan is a clan (based on the definable contact aCb iff $a, b \not\vdash 0$).

Denote for $a \in B$

$H(a) = \{\Gamma \in SCLANS(B) : a \in \Gamma\}$,

$h(a) = \{U \in Ult(B) : a \in U\}$.

Define a topology in the set SCLANS(B) having the set $\{H(a) : a \in B\}$ as a basis for the closed sets.

Lemma 1.

1. If $\Gamma \in SCLANS(B))$ and $a \in \Gamma$, then $(\exists U \in Ult(\Gamma))(a \in U \subseteq \Gamma)$,

2. $a_1, \dots, a_n \not\vdash b$ iff $(\exists \Gamma \in SCLANS(B))$ $(\{a_1, \dots, a_n\} \subseteq \Gamma$ and $b \notin \Gamma)$.

3. $a_1, \dots, a_n \not\vdash b$ iff $(\exists \Gamma \in SCLANS(B))(\exists U_1 \dots U_n \in Ult(\Gamma))$ $(a_1 \in U_1, \dots, a_n \in U_n$ and $b \notin \Gamma)$,

Lemma 2. (i) $H(a+b) = H(a) \cup H(b)$,

(ii) $H(a^*) = Cl - H(a)$,

(iii) $a \leq b$ iff $H(a) \subseteq H(b)$,

(iv) $a_1, \ldots a_n \vdash b$ iff $H(a_1) \cap \ldots, \cap H(a)_n \subseteq H(b)$.

(v) The topology in SCLANS(B) is semiregular, T_0 and compact,

(vi) H is a dense embedding of $\underline{B}$ into the S-algebra $RC(SCLANS(B))$, If $\underline{B}$ is a complete Boolean algebra then H is a isomorphism onto $RC(SCLANS(B))$.

Theorem 1. (Topological representation of S-algebras.) Every S-algebra can be densely embedded into the S-algebra $RC(X)$ of regular closed subsets of some semiregular, compact T_0 space X. If $\underline{B}$ is complete then $\underline{B}$ is isomorphic with $RC(X)$. (Proof. By Lemma 1 and Lemma 2.)

Lemma 3.(i) $h(a+b) = h(a) \cup (b)$,

(ii) $h(a^*) = -h(a)$, $h(a \cap b) = h(a) \cap h(b)$,

(iii) $a \leq b$ iff $h(a) \subseteq h(b)$,

(iv) Let X be the set of ultrafilters of $\underline{B}$ and $Y = \{Ult(\Gamma) : \Gamma \in SCLANS(B)\}$. Then

$a_1, \ldots a_n \vdash b$ iff $h(a_1), \ldots h(a_n) \vdash_{discr} h(b)$.

Theorem 2.(Discrete representation of S-algebras.) Every S-algebra $\underline{B}$ can be embedded into the discrete S-algebra of some discrete S-space (X, Y).

Proof. Apply Lemma 1(iii) and Lemma 3.

S-ALGEBRAS WITH ADDITIONAL AXIOMS

1. The axiom of connectedness can be formulated as follows:

(Con) If $a \neq 0$ and $a \neq 1$ then $a, a^* \nvdash 0$

Every connected S-algebra can be repesented in a connected topological space.

2. T_1 axiom: Let A be a finite subset of $\underline{B}$. If $A \nvdash b$, then there exists a finite subset B such that $A, B \nvdash 0$ and $B, b \vdash 0$.

T_1 S-algebras are representable in T_1 topological spaces. Abstract points in $\underline{B}$ are maximal S-clans.

3. T_2 axiom. It is a conjunction of the following two axioms:

(T_1') If $A \nvdash a$, then $(\exists b)(A, b \nvdash 0$ and $a, b \vdash 0$,

(Nor') If $A, b \vdash 0$ then $(\exists c, d)(c + d = 1$ and $A, c \vdash 0$ and $b, d \vdash 0)$.

Note that (T_1') implies (T_1) and that (Nor') implies (Nor) (with definable contact aCb iff $a, b \nvdash 0)$.

Axiom T_2 is true in all compact Hausdorff (T_2) spaces.

To represent T_2 S-algebras we need a new kind of abstract points, called S-clusters.

Definition (S-clusters). An S-clan Γ is called S-cluster if it satisfies the condition

(S-cluster) If $a \notin \Gamma$ then $(\exists b \in \Gamma)(a, b \vdash 0)$.

The notion of S-cluster generalizes the notion of cluster from the theory of contact algebras and proximity spaces.

Every S-cluster is a maximal S-clan. Axiom (Nor') implies that every maximal S-clan is an S-cluster.

Let $\underline{B}$ satisfies T_2 and let the abstract points of $\underline{B}$ are the S-clusters. Then the topology of the set of all S-clusters is compact and Hausdorff (T_2). Every such an S-algebra can be embedded into the S-algebra $RC(X)$ of a compact Hausdorff semiregular space X.

PROPOSITIONAL LOGICS BASED ON S-ALGEBRAS – S-LOGICS.

1. **The language of the minimal S-Logic $\mathbb{L}_{min}$.**

- A denumeral set of Boolean variables,
- Boolean constants $0, 1$,
- Boolean operations: $+, ., *$,
- Relational symbols $\leq$, $\vdash$,
- Propositional connectives: $\vee, \wedge, \Rightarrow, \Leftrightarrow, \neg$, and the constants $\bot, \top$.

2. Boolean terms are defined in a standard way from Boolean variables, Boolean constants by means of Boolean operations.

3. Formulas.

• Atomic formulas: propositional constants, $(a \leq b)$, where a, b are Boolean terms, and $(A \vdash b)$, where A is a finite set (including the empty set) of Boolean terms and b is a Boolean term. (here we adopt all abbreviations for the use of sequents)

• Formulas are defined in a standard way from atomic formulas by means of propositional connectives.

4. Semantics in S-algebras.

Let $\underline{B} = (B, \vdash)$ be an S-algebra. A function v from the set of Boolean variables into B is called a valuation if it satisfies the conditions: $v(0) = 0$, $v(1) = 1$. v is then extended inductively in a homomorphic way to the set of Boolean terms. If $A = \{a_1, \ldots, a_n\}$ is a non-empty finite set of Boolean terms then $v(A) =_{def} \{v(a_1), \ldots, v(a_n)\}$, $v(\varnothing) =_{def} \{1\}$.

A pair $M = (\underline{B}, v)$ is called a model. The truth for a formula α in $(\underline{B}, v)$ (denoted $(\underline{B}, v \models \alpha)$ is defined as follows:

- $(\underline{B}, v) \models \top$, $(\underline{B}, v) \not\models \bot$,
- $(\underline{B}, v) \models a \leq b$ iff $v(a) \leq v(b)$,
- $(\underline{B}, v) \models A \vdash b$ iff $v(A) \vdash v(b)$.

A formula α is true in an S-algebra $\underline{B}$ if it is true in all models over $\underline{B}$; α is true in a class Σ of S-algebras if it is true in every S-algebra from Σ.

5. Axiomatization of $\mathbb{L}_{min}$

Axioms

• Any set of axiom schemes of classical propositional calculus

• Any set of axiom schemes of Boolean algebra written in terms of Boolean order $\leq$ (example: $(a \leq b) \wedge (b \leq c) \Rightarrow (a \leq c)$, $a.b \leq a$, etc).

• Axioms for $\vdash$ – adapting the algebraic axioms of S-algebra, examples: $a \vdash b \Leftrightarrow a \leq b$, $(A, a \vdash c) \wedge (A, b \vdash c) \Rightarrow (A, a + b \vdash c)$, etc.

Rules of inference: modus ponens(MP) $\frac{\alpha, \alpha \Rightarrow \beta}{\beta}$

The notion of a theorem is defined in a standard way.

6. Completeness theorem for $\mathbb{L}_{min}$

Theorem. (I) (Weak form)

The following conditions are equivalent for any formula α:

(i) α is a theorem of $\mathbb{L}_{min}$,

(ii) α is true in the class of all S-algebras,

(iii) α is true in all topological S-algebras,

(iv) α is true in all topological S-algebras over semiregular, compact T_0 topological space,

(v) α is true in all discrete S-algebras.

(II) (Strong form)

The following conditions are equivalent for any set Γ of formulas:

(i) Γ is consistent,

(ii) Γ has a model in the class of all S-algebras

in all topological S-algebras,

(iv) Γ has a model in all topological S-algebras over semiregular, compact T_0 topological space,

(v) Γ has a model in all discrete S-algebras.

Proof(idea) - by an adaptation of the Henkin proof for the first-order logic

7. Finite model property and decidability

Theorem. The following conditions are equivalent for any formula α:

(i) α is a theorem of $\mathbb{L}_{min}$,

(ii) α is true in all finite S-algebras with cardinality 2^{2^n}, where n is the number of Boolean variables occurring in α.

Proof (idea). If α is not a theorem, then it is falsified in some S-algebra $\underline{B}$. Take the Boolean subalgebra of $\underline{B}$ generated by the value of the Boolean variables occurring in α.

S-logics with additional axioms and rules

(Con) Axiom of connectedness

$a \nvdash 0 \wedge 1 \nvdash a \Rightarrow a, a^* \nvdash 0$

The logic $\mathbb{L}_{min} + (Con)$ is sound and complete in S-algebras over connected topological spaces (the space can be chosen to be semiregular, compact and T_0). It has fmp and hence is decidable.

T_1-rule.

If we want to obtain an S-logic complete in all S-algebras over T_1-topological spaces, we have to extend the logic $\mathbb{L}_{min}$ with the following ∞-rule:

Let A be a finite set of Boolean terms, a be a Boolean term and $b_1, b_2, \ldots$ be a denumerable sequence of different Boolean variables not occurring in A and a and let $B_n = \{b_1, \ldots, b_n\}$, $n = 0, 1, \ldots$.

(T$_1$-rule) $\frac{B_n, a \vdash 0 \Rightarrow A, B_n \vdash 0, n=0,1,\ldots}{A \vdash a}$

This logic is strongly complete in the class of all S-algebras over semiregular, compact and T_1 topological spaces. If we add the axiom of connectedness then the spaces are connected.

T_2-rules

If we want to obtain an S-logic complete in the class of all S-algebras over compact Hausdorff spaces, we have to extend the logic $\mathbb{L}_{min}$ with the following two rules:

(T_1^+-rule) $\frac{a,b\vdash 0 \Rightarrow A,b\vdash 0}{A\vdash a}$, where b is a Boolean variable not occurring in A and a.

(Nor$^+$-rule) $\frac{(A,b\nvdash 0)\vee(a,b^*\nvdash 0)}{A,a\nvdash 0}$, where b is a Boolean variable not occurring in A and a.

This logic is strongly complete in the class of all S-algebras over semiregular and compact Hausdorff spaces. If we add the axiom of connectedness then the spaces are connected.

END