A multimodal logic for closeness

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Qualitative Reasoning (QR)

- QR is very useful for searching solutions to problems about the behavior of physical systems without using differential equations or exact numerical data.
- It is possible to reason about incomplete knowledge by providing an abstraction of the numerical values.
- QR has applications in AI, such as Robot Kinematics, Data Analysis, and dealing with movements.

Order of Magnitude QR

- A partition of the real line in qualitative classes (small, medium, large,...) is considered. The absolute approach.
- A family of binary order of magnitude relations which establishes different comparison relations (negligibility, closeness, comparability, ...). The relative approach.
- We have defined some logics which bridge the absolute and relative approaches.

Preliminary definitions

We will consider a strictly ordered set of real numbers $(\mathbb{S}, <)$ divided into the following qualitative classes:

$$\mathsf{NL} = (-\infty, -\gamma) \qquad \mathsf{PS} = (+\alpha, +\beta]$$

$$\mathsf{NM} = [-\gamma, -\beta) \qquad \mathsf{INF} = [-\alpha, +\alpha] \qquad \mathsf{PM} = (+\beta, +\gamma]$$

$$\mathsf{NS} = [-\beta, -\alpha) \qquad \mathsf{PL} = (+\gamma, +\infty)$$

Note that all the intervals are considered relative to $\mathbb{S}$.
We will consider each qualitative class to be divided into disjoint intervals called *proximity intervals*, as shown in the figure below. The qualitative class INF is itself a proximity interval.

Figure: Proximity intervals.

Preliminary definitions

Definition

Let $(\mathbb{S}, <)$ be a linearly ordered set divided into the qualitative classes above:

- A *proximity structure* is a finite set $\mathcal{I}(\mathbb{S}) = \{I_1, I_2, \dots, I_n\}$ of intervals in $\mathbb{S}$, such that:
  1. For all $I_i, I_j \in \mathcal{I}(\mathbb{S})$, if $i \neq j$, then $I_i \cap I_j = \varnothing$.
  2. $I_1 \cup I_2 \cup \cdots \cup I_n = \mathbb{S}$.
  3. For all $x, y \in \mathbb{S}$ and $I_i \in \mathcal{I}(\mathbb{S})$, if $x, y \in I_i$, then x, y belong to the same qualitative class.
  4. $\mathsf{INF} \in \mathcal{I}(\mathbb{S})$.
- Given a proximity structure $\mathcal{I}(\mathbb{S})$, the binary relation of closeness $\mathfrak{c}$ is defined, for all $x, y \in \mathbb{S}$, as follows: $x \,\mathfrak{c}\, y$ if and only if there exists $I_i \in \mathcal{I}(\mathbb{S})$ such that $x, y \in I_i$.

The language $\mathcal{L}(MQ)^{\mathcal{P}}$

Introducing the Syntax

Modal connectives $\overrightarrow{\Box}$ and $\overleftarrow{\Box}$ to deal with the usual ordering $<$.

The modal operator $\boxed{c}$ will be used to represent closeness.

Their informal meanings are the following

- $\overrightarrow{\Box}A$ means *A is true in every point greater than the current one*.
- $\overleftarrow{\Box}A$ means *A is true in every point smaller than the current one*.
- $\boxed{c}A$ means *A is true in every point close to the current one*

The language $\mathcal{L}(MQ)^{\mathcal{P}}$

Syntax

The formulas are defined as follows:

$$A = p \mid \xi \mid c_i \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid \overrightarrow{\Box} A \mid \overleftarrow{\Box} A \mid \boxed{c} A$$

where

- p represents the propositional variables
- ξ is a metavariable denoting any milestone $\alpha^-, \alpha^+, \beta^-, \beta^+, \gamma^-, \gamma^+$
- c_i are proximity constants (finitely many)
- The connectives $\neg, \wedge, \vee$ and $\to$ are the classical ones
- $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\boxed{c}$ are the previous unary modalities

We will also introduce abbreviations for qualitative classes, for instance, 'ps' stands for $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}\beta^+) \vee \beta^+$.

Definition

A *frame* for $\mathcal{L}(MQ)^{\mathcal{P}}$ is a tuple $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P})$, where:

1. $(\mathbb{S}, <)$ is a strict linearly ordered set.
2. $\mathcal{D} = \{+\alpha, -\alpha, +\beta, -\beta, +\gamma, -\gamma\}$ is a set of designated points in $\mathbb{S}$ (called *frame constants*).
3. $\mathcal{I}(\mathbb{S})$ is a proximity structure.
4. $\mathcal{P}$ is a bijection (called *proximity function*), $\mathcal{P}: \mathcal{C} \longrightarrow \mathcal{I}(\mathbb{S})$, that assigns to each proximity constant c a proximity interval.

The language $\mathcal{L}(MQ)^{\mathcal{P}}$

Semantics

Definition

Let Σ be a frame for $\mathcal{L}(MQ)^{\mathcal{P}}$, an *MQ- model* is an ordered pair $\mathcal{M} = (\Sigma, h)$, where h is a *meaning function* (or, *interpretation*) $h\colon \mathcal{V} \longrightarrow 2^{\mathbb{S}}$.
Any interpretation can be uniquely extended to the set of all formulas in $\mathcal{L}(MQ)^{\mathcal{P}}$ (also denoted by h) as follows:

$$
\begin{aligned}
h(\overrightarrow{\Box} A) &= \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } x < y\} \\
h(\overleftarrow{\Box} A) &= \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } y < x\} \\
h(\boxed{c} A) &= \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } x\, \mathfrak{c}\, y\} \\
h(\alpha^{+}) &= \{+\alpha\} \qquad h(\beta^{+}) = \{+\beta\} \qquad h(\gamma^{+}) = \{+\gamma\} \\
h(\alpha^{-}) &= \{-\alpha\} \qquad h(\beta^{-}) = \{-\beta\} \qquad h(\gamma^{-}) = \{-\gamma\} \\
h(c_i) &= \{x \in \mathbb{S} \mid x \in \mathcal{P}(c_i)\}
\end{aligned}
$$

The definitions of *truth*, *satisfiability* and *validity* are the usual ones.

An axiom system for $\mathcal{L}(MQ)^{\mathcal{P}}$

The axiom system $MQ^{\mathcal{P}}$ consists of all the tautologies of classical propositional logic plus the following axiom schemata and rules of inference:

For white connectives

K1 $\overrightarrow{\Box}(A \to B) \to (\overrightarrow{\Box}A \to \overrightarrow{\Box}B)$

K2 $A \to \overrightarrow{\Box}\overleftarrow{\Diamond}A$

K3 $\overrightarrow{\Box}A \to \overrightarrow{\Box}\overrightarrow{\Box}A$

K4 $\left(\overrightarrow{\Box}(A \vee B) \wedge \overrightarrow{\Box}(\overrightarrow{\Box}A \vee B) \wedge \overrightarrow{\Box}(A \vee \overrightarrow{\Box}B)\right) \to \left(\overrightarrow{\Box}A \vee \overrightarrow{\Box}B\right)$

For frame constants

c1 $\overleftarrow{\Diamond}\xi \vee \xi \vee \overrightarrow{\Diamond}\xi$

c2 $\xi \to (\overleftarrow{\Box}\neg\xi \wedge \overrightarrow{\Box}\neg\xi)$

c3 $\gamma^- \to \overrightarrow{\Diamond}\beta^-$

c4 $\beta^- \to \overrightarrow{\Diamond}\alpha^-$

c5 $\alpha^- \to \overrightarrow{\Diamond}\alpha^+$

c6 $\alpha^+ \to \overrightarrow{\Diamond}\beta^+$

c7 $\beta^+ \to \overrightarrow{\Diamond}\gamma^+$

An axiom system (cont'd)

For proximity constants (for all $i, j \in \{1, \dots, r\}$)

p1 $\bigvee_{i=1}^{r} c_i$

p2 $c_i \to \neg c_j$ (for $i \neq j$)

p3 $(\overleftarrow{\Diamond} c_i \wedge \overrightarrow{\Diamond} c_i) \to c_i$

p4 $\overleftarrow{\Diamond} c_i \vee c_i \vee \overrightarrow{\Diamond} c_i$

Mixed axioms (for all $i \in \{1, \dots, r\}$)

m1 $(c_i \wedge \mathrm{qc}) \to \left(\overleftarrow{\Box}(c_i \to \mathrm{qc}) \wedge \overrightarrow{\Box}(c_i \to \mathrm{qc})\right)$

m2 $(c_i \wedge \mathrm{inf}) \to \left(\overleftarrow{\Box}(\mathrm{inf} \to c_i) \wedge \overrightarrow{\Box}(\mathrm{inf} \to c_i)\right)$

m3 $\boxed{\scriptstyle c} A \leftrightarrow \left(A \wedge \bigvee_{i=1}^{r} \left(c_i \wedge (\overleftarrow{\Box}(c_i \to A) \wedge \overrightarrow{\Box}(c_i \to A))\right)\right)$

m4 $\boxed{\scriptstyle n} A \leftrightarrow \left(\left(\mathrm{inf} \to (\overleftarrow{\Box}(\neg\mathrm{inf} \to A) \wedge \overrightarrow{\Box}(\neg\mathrm{inf} \to A))\right) \wedge \left((\mathrm{ns} \vee \mathrm{ps}) \to (\overleftarrow{\Box}(\mathrm{nl} \to A) \wedge \overrightarrow{\Box}(\mathrm{pl} \to A))\right)\right)$

An axiom system (cont'd)

Rules of inference:

- **(MP)** Modus Ponens for $\rightarrow$.
- **(N$\overrightarrow{\Box}$)** If $\vdash A$ then $\vdash \overrightarrow{\Box} A$.
- **(N$\overleftarrow{\Box}$)** If $\vdash A$ then $\vdash \overleftarrow{\Box} A$.

The syntactical notions of *theoremhood* and *proof* for $MQ^{\mathcal{P}}$ are as usual.

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Theorem (Completeness)

If A is valid formula of $\mathcal{L}(MQ)^{\mathcal{P}}$, then A is a theorem of $MQ^{\mathcal{P}}$.

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Theorem (Decidability)

$MQ^{\mathcal{P}}$ is decidable.

Once upon a time . . .

An anecdote in the early nineties

Luis and . . . (yes it's me)

Once upon a time . . .

An anecdote in the early nineties

Luis and . . . (yes it's me)

. . . having some tapas

Once upon a time . . .

An anecdote in the early nineties

. . . after some time, logically,

Once upon a time . . .

An anecdote in the early nineties

. . . after some time, logically,

(the waitress tries to take the plate)

Once upon a time . . .

An anecdote in the early nineties


(the waitress tries to take the plate)
and Luis said . . .

Once upon a time . . .

An anecdote in the early nineties


(the waitress tries to take the plate)
and Luis said . . .

"Please, leave it in the table, . . .
and bring some bread to soak up the sauce !!"

A little present for Luis (in Spanish)

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Lentamente analiza los problemas
usando asaz ideas novedosas,
igual que al tratar mundanas cosas,
siempre sale airoso de un dilema

fabricando unos modales teoremas,
amalgama de las formas más hermosas.

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Resuelve los entuertos con sus prosas
impulsando al IRIT, del que es emblema.

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Ñoñerías aparte, a duermevela
atraviesa la senda serpenteante
subiendo una científica montaña,

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del Cerro es su apellido (tiene tela),
interpretando a un caballero andante
el de La Mancha, Toledo, España

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