Mixed Algebras and their Logics

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Dedicated to Luis Fariñas del Cerro

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Necessity and sufficiency — a short history

Possibility and sufficiency (PS) algebras

Mixed algebras (MIA)

Weak mixed algebras (wMIA)

The logic $K^{\sim}$

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The common form of a mathematical theorem consists in that "the truth of some properties for some objects is *necessary* and/or *sufficient* condition for other properties to hold for other objects". To formalize this, one happens to resort to Kripke modal logic K which, having in the syntax the notions of 'property' and 'necessity', appears to provide a reliable metamathematical fundament.

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But, what about formalizing of the 'sufficiency'?
The first and trivial attempt is to grammatically reduce the 'sufficiency' to 'necessity' saying that
"x is sufficient for p" iff "p is necessary in x".
Evidently this will not enrich our knowledge.

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In 1985–1987 several people from Sofia modal logic group studied the bimodal logic $K^{\sim}$ of $\Box$ and $\boxbar$ and a lot of its extensions.

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In 2000 Ivo Düntsch and Ewa Orłowska comming from information systems introduced and studied in Jó nsson–Tarski style so-called *mixed algebras*, (MIA). Now we continue these studies.

Possibility and sufficiency (PS) algebras

Let B be a Boolean algebra. A *possibility operator* on B is a normal and additive function $f : B \to B$, i.e. $f(0) = 0$ and $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in B$. The dual operator of a possibility operator is called *necessity operator*.

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A *sufficiency operator* on B is a co-normal and co-additive function $g : B \to B$, i.e. $g(0) = 1$ and $g(a \vee b) = g(a) \wedge g(b)$ for all $a, b \in B$.

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The structure (B, f, g) is called *PS*-algebra if B is a Boolean algebra, f and g are possibility and sufficiency operators respectively.
In every *PS*-algebra the mapping u defined by
$u(a) = f^{\partial}(a) \wedge g^*(a)$ for all $a \in B$ is a necessity operator on B.

Mixed algebras (MIA)

A mixed algebra (MIA) is a PS-algebra is a PS-algebra (B, f, g) such that its canonical frame $(\mathrm{Ult}(B), R_f, R_g)$ satisfies the condition $R_f = R_g$, where
$FR_f G \leftrightarrow F \in \bigcap\{h(f(x)) \mid x \in G\}$ and
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In a previous paper Düntsch and Orłowska proved that the class of all MIA, **MIA**, is not first-order axiomatizible.

Weak mixed algebras (wMIA)

A *weak mixed algebra* (wMIA) is a *PS*-algebra is a *PS*-algebra (B, f, g) such that

$$a \wedge b \neq 0 \to g(a) \leq f(b) \qquad \text{for all } a, b \in B.$$

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Theorem. Let (B, f, g) be a *PS*-algebra. Then
B is a wMIA iff $u(a) = 1$ if $a = 1$ and $u(a) = 0$ otherwise.

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Theorem. Let (B, f, g) be a *PS*-algebra. Then
B is a wMIA iff $u(a) = 1$ if $a = 1$ and $u(a) = 0$ otherwise.

Thus, each wMIA is a discriminator algebra
$(t(a) = u^{\partial}(a) = \neg u(\neg a)$ is a discriminator term).

The logic $K^{\sim}$

To the logic $K^{\sim}$ correspond the PS-algebras satisfying S5 conditions for u. Namely,
$u(a) \leq a, \;\; u(a) \leq u(u(a)), \;\; a \leq u(u^{\partial}(a))$ for all $a \in B$.
Let **KMIA** be the class of all PS-algebras satisfying S5 conditions for u.
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Theorem. $\mathbf{Eq}(\mathbf{wMIA}) = \mathbf{KMIA}$.

Thank you!