On the relation between possibilistic logic and modal logics of belief

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Outline

1. Possibilistic vs. modal logic
2. Minimal Epistemic Logic
3. Generalized Possibilistic Logic
4. GPL with objective formulas

Possibility theory

A formalism for representing uncertainty due to incomplete information

- Incomplete information modelled by (fuzzy) subsets of mutually exclusive values of a quantity (or possible worlds)
- **Possibility distributions** $\pi : \Omega \to [0, 1]$: $\pi(w)$ is the degree of possibility that w is the actual value or world
- $\max \pi = 1$ (consistency)
- Two set functions similar to probability functions
  - **Possibility measure**: $\Pi(A) = \max_{w \in A} \pi(w)$ (plausibility)
  - **Necessity measure**: $N(A) = 1 - \Pi(\overline{A})$ (certainty)

A proposition can be more or less impossible ($\Pi < 1$), more or less certain $N > 0$, or unknown ($N = 0, \Pi = 1$).

Possibility theory : previous works

- Shackle (1949 on), English economist. Degrees of potential surprize on a surprize scale
- Lewis (1973 on): Comparative possibility relations and their modal logics for counterfactuals
- Zadeh (1978) : imprecise linguistic statements modelled by fuzzy sets interpreted as possibility distributions
- Spohn (1988): degrees of disbelief on the scale of integers

The only numerical representations of Lewis comparative relations are possibility measures (Dubois 1986)

KD Modal logic and possibility theory: analogy

	Possibility theory	Modal logic
Tools	set functions N, Π	modalities $\Box, \Diamond$
Scale	[0, 1]	$\{0, 1\}$
Adjunction	$N(\phi \wedge \psi) = \min(N(\phi), N(\psi))$	$\Box(\phi \wedge \psi) \equiv \Box\phi \wedge \Box\psi$
Duality	$\Pi(\phi) = 1 - N(\neg\phi)$	$\Box\phi \equiv \neg\Diamond\neg\phi$
	$\Pi(\phi) \geq N(\phi)$	$\Box\phi \to \Diamond\phi$

It is natural to equate $\Box\phi$ and $N(\phi) > 0$

Earlier connections between possibility theory and modal logic

- Fariñas del Cerro and Prade (1986): possibility theory, incomplete information databases and the modal logic of rough sets
- Dubois, Prade, Testemale (1988): Accessibility relation representing relative specificity between epistemic states
- Fariñas del Cerro and Herzig (1991): Possibility theory and Lewis modal logics using comparative possibility
- Boutilier (1994): interprets a possibility relation as an accessibility relation between possible worlds
- Esteva Godo Hajek (1995): Casting uncertainty theories in the language of fuzzy modal logics with Kripke semantics
- Resconi Klir etc. (1992-95): Relating degrees of uncertainty to accessibility relations
- Halpern, Ognjanovic, etc.

Elementary possibilistic logic

Possibility theory led to possibilistic logic (Dubois Lang Prade, 1987).

Syntax : Poslog formulas are

- Pairs (ϕ, a) where ϕ is a propositional formula in PROP and $a \in (0, 1]$.
- A poslog base B is a conjunction of such pairs (ϕ_i, a_i).

Intended meaning : $N(\phi) \geq a$.

- **Axioms** : $(\phi, 1)$ for PROP tautologies ϕ.
- **Basic inference rules** (justified by the laws of possibility theory)
  - Resolution : $(\phi \vee \psi, a); (\neg\phi \vee \chi, b) \vdash (\psi \vee \chi, \min(a, b))$
  - Weight weakening : If $a \geq b$ then $(\phi, a) \vdash (\phi, b)$
- *Inconsistency degree* : $Inc(B) = \max\{a : B \vdash (\bot, a)\}$.
- *Nontrival, non-monotonic consequences of B* : ϕ s.t. $B \vdash (\phi, a)$, with $a > Inc(B)$.

Possibilistic logic and Modal logic KD

	PosLog	Modal logic
Atoms	$(\phi, a), \phi \in PROP, a \in (0, 1]$	PROP atoms
Connectives	$\wedge$	$\wedge, \neg, \Box$
Modalities	No nesting	Nested modalities
Properties	$(\phi \wedge \psi, a) \equiv (\phi, a) \wedge (\psi, a)$	$\Box(\phi \wedge \psi) \equiv \Box\phi \wedge \Box\psi$
Semantics	possibility distributions	accessibility relations

So

- possibilistic logic is a graded belief logic with a very poor syntax
- modal logic can model all-or-nothing combinations of beliefs in a more expressive syntax.
- Restricted to formulas $(p, 1)$, PosLog is isomorphic to PROP

A minimal two-tiered epistemic logic (MEL)

How to construct a modal logic with possibilistic semantics?
Idea: Find the minimal language to express the statement that a proposition is unknown, encoding a belief $N(\phi) = 1$ as $\Box\phi$.

1. Standard propositional Boolean logic language $\mathcal{L}$
   - Propositional variables $\mathcal{V} = \{a, b, c, \ldots, p, \ldots\}$
   - $\phi, \psi, \ldots$ propositional formulae of $\mathcal{L}$ built using conjunction, disjunction, and negation $(\wedge, \vee, \neg)$
2. Upper level: A propositional language $\mathcal{L}_\Box$
   - Variables: $\mathcal{V}_\Box = \{\Box\phi : \phi \in \mathcal{L}\}$
   - $\mathcal{L}_\Box$ propositional language based on $\mathcal{V}_\Box$

$\Rightarrow$ The "subjective" fragment of KD (or S5) without modality nesting.

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The MEL axioms

$\mathcal{L}_\Box$ is the minimal language to express partial knowledge about the truth of propositions. (you can write "the agent ignores ϕ" as $\neg\Box\phi \wedge \neg\Box\neg\phi$)

Axioms

(PL) Axioms of PROP for $\mathcal{L}_\Box$-formulas

(K) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

(D) $\Box\phi \to \Diamond\phi$

(Nec) $\Box\phi$, for each $\phi \in \mathcal{L}$ that is a PROP tautology, i.e. if $Mod(\phi) = \Omega$.

the **inference rule** is modus ponens.

$$B \vdash_{MEL} \Phi \text{ if and only if } B \cup \{K, D, Nec\} \vdash_{PROP} \Phi$$

Note : in KD45, Nec is an inference rule (necessitation).

Possibilistic semantics

The semantics does not require accessibility relations

- $N(\phi) = 1$ means that ϕ holds in all worlds considered possible by the agent, i.e., there is a non-empty set E of possible interpretations (the epistemic state of the agent) such that $E \subseteq [\phi]$.
- The epistemic models of $\Box\phi$ are $\{E \neq \emptyset : E \subseteq [\phi]\} \subseteq 2^{\Omega}$

Satisfiability

- $E \models \Box\phi$ if $E \subseteq [\phi]$ (ϕ is certainly true in the epistemic state E)
- $E \models \Phi \wedge \Psi$ if $E \models \Phi$ and $E \models \Psi$
- $E \models \neg\Phi$ if $E \models \Phi$ is false

MEL is sound and complete with respect to this semantics

Clue: an epistemic model of Φ is a standard propositional interpretation of $\mathcal{L}_{\Box}$.

MEL is just a propositional logic

A fragment of KD45, etc., with a restricted language but...

- *MEL* does NOT allow for (non-modal) propositional formulas : The languages $\mathcal{L}$ and $\mathcal{L}_\Box$ are disjoint.
  - KD45 axioms (**4**: $\Box\Phi \to \Box\Box\Phi$; **5**: $\neg\Box\Phi \to \Box\neg\Box\Phi$) cannot be written in MEL.
- In MEL, formulas are evaluated on epistemic states ($E \models \Box\phi$) while in KD45 formulas are evaluated on possible worlds ($w \models \Box\phi$) via accessibility relations
- KD45 simplifies the expressions in KD, MEL minimally augments the expressive power of PROP.
- MEL has the deduction theorem, KD45 has not always.
- KD45 accounts for introspection: MEL describes what an agent knows about the epistemic state of another agent

Positioning MEL wrt. Agent-based reasoning

Observer	$\longleftarrow$	Agent	$\longleftarrow$	World
Belief about Agent		Belief about world		Actual world
$\mathcal{E} \subseteq 2^{\Omega}$		$E \subseteq \Omega$		$w \in \Omega$
MEL		PROP		

- E is the set of worlds considered possible by the agent
- $\mathcal{E}$ is the set of epistemic states of the agent considered possible by the observer
- E is represented by a PROP base, $\mathcal{E}$ by a MEL base

Generalized Possibilistic Logic: MEL + Poslog

Syntax : GPL formulas use graded KD modalities and form a language $\mathcal{L}^k_\Box$ using a scale $\Lambda_k = \{0, \frac{1}{k}, \frac{2}{k}, ..., 1\}$.

- Atoms : $\Box_a \phi$ where ϕ is a propositional formula and $a \in \Lambda^+_k = \{\frac{1}{k}, \frac{2}{k}, ..., 1\}$. They stand for (ϕ, a), i.e. $N(\phi) \geq a$.
- All propositional formulas from atoms $\Box_a(\phi)$.

we can express : $\Pi(\phi) \geq \frac{i}{k}$, as $\neg\Box_{1-\frac{i-1}{k}}(\neg\phi)$

Axioms

(PL) Axioms of PROP for *GPL*-formulas

(K) $\Box_a(\phi \to \psi) \to (\Box_a\phi \to \Box_a\psi)$

(D) $\Box_a\psi \to \neg\Box_b\neg\psi$

(Nec) $\Box_a\phi$, for each tautology $\phi \in \mathcal{L}$

(W) $\Box_a\phi \to \Box_b\phi$, if $a \geq b$

If $a = b$ is fixed, we get a copy of MEL.

Generalized Possibilistic Logic : Semantics and completeness

The semantics uses gradual epistemic models

- $\vdash \Box_a \phi$ means that $N(\phi) \geq a$
  computed from possibility distribution π on Ω.
  (ϕ is certainly true at level at least a in the epistemic state π)
- The epistemic models of $\Box_a \phi$ are $\{\pi : \min_{w \not\models \phi} 1 - \pi(w) \geq a\}$

Satisfiability

- $\pi \models \Box_a \phi$ if $N(\phi) \geq a$
- $\pi \models \Phi \wedge \Psi$ if $\pi \models \Phi$ and $\pi \models \Psi$
- $\pi \models \neg\Phi$ if $\pi \models \Phi$ is false

GPL is sound and complete with respect to this semantics

Clue: an epistemic model of Φ is a standard propositional interpretation of $\mathcal{L}^k_\Box$.

Positioning GPL wrt. Agent-based reasoning

Observer	$\longleftarrow$	Agent	$\longleftarrow$	World
Knowledge about Agent		Knowledge about world		Actual world
$\mathcal{E} \subseteq \Lambda^{\Omega}$		$\pi : \Omega \to \Lambda$		$w \in \Omega$
A set of π's		A possibility distribution		
GPL		PosLog		

Extending GPL to reason about the actual world and someone's beliefs

- Extended language $\mathcal{L}_{\Box}^{k+}$ of GPL^{+} with objective formulas
  - If $\phi \in \mathcal{L}$, then $\phi \in \mathcal{L}_{\Box}^{k+}$
  - If $a \in \Lambda^k \setminus \{0\}$, then $\Box_a \phi \in \mathcal{L}_{\Box}^{k+}$
  - If $\Phi, \Psi \in \mathcal{L}_{\Box}^{k+}$ then $\neg\Phi, \Phi \wedge \Psi \in \mathcal{L}_{\Box}^{k+}$
- Semantics for GPL^{+}: "pointed" GPL epistemic models, i.e., structures (w, π), where $w \in \Omega$ and $\pi \in (\Lambda^k)^\Omega$.
- Truth-evaluation rules of formulas of $\mathcal{L}_{\Box}^{k+}$ in (w, π):
  - $(w, \pi) \models \phi$ if $w \models \phi$, as $\phi \in \mathcal{L}$
  - $(w, \pi) \models \Box_a \phi$ if $N(\phi) \geq a$ in π.
  - usual rules for $\neg$ and $\wedge$ on $\Phi \in \mathcal{L}_{\Box}^{k+}$.
- Logical consequence, as usual: $\Gamma \models \Phi$ if, for every structure (w, π), $(w, \pi) \models \Phi$ whenever $(w, \pi) \models \Psi$ for all $\Psi \in \Gamma$.

Completeness of GPL$^+$

Axiomatic system : We use the same axioms and inference rule as GPL (only language and semantics change).

Lemma

$\Gamma \vdash_{GPL^+} \Phi$ *iff*
$\Gamma \cup \{\Box_1 \phi \mid \vdash_{PROP} \phi\} \cup \{$*instances of axioms (K), (D) (W)* $\} \vdash_{PROP} \Phi$

Theorem

$\Gamma \vdash_{GPL^+} \Phi$ *iff* $\Gamma \models \Phi$ *under the pointed e-model semantics.*

We get closer to S5 if we add axiom T: $\Box_a \phi \to \phi$, which restricts pointed e-models to (w, π) where $w \in \{w : \pi(w) > 1 - a\}$ (GPL^{+T}).

Relating MEL and MEL$^+$ to KD45 and S5

MEL$^+$ is the restriction of GPL$^+$ to $a = 1$.
(models are pointed e-models (w, E))

MEL^{+T} is MEL$^+$ with axiom T ($\Box\phi \to \phi$)
(models are pointed e-models (w, E) with $w \in E$.)

Theorem

Let Φ be a formula from $\mathcal{L}_\Box$. Then

- *$MEL \vdash \Phi$ iff $L \vdash \Phi$ for $L \in \{KD, KD4, KD45, S5\}$.*

Let Φ be a formula from $\mathcal{L}_\Box^+$. Then

- *$MEL^+ \vdash \Phi$ iff $L \vdash \Phi$ for $L \in \{KD, KD4, KD45\}$.*
- *$MEL^{+T} \vdash \Phi$ iff $S5 \vdash \Phi$*

Relating MEL and MEL$^+$ to KD45 and S5

Since any formula of KD45 and S5 is logically equivalent to another formula without nested modalities:

Theorem

The following conditions hold true:

- *For any arbitrary modal formula Φ, there is a formula $\Phi' \in \mathcal{L}^+_\Box$ such that KD45 $\vdash \Phi$ iff MEL$^+ \vdash \Phi'$.*
- *For any arbitrary modal formula Φ, there is a formula $\Phi' \in \mathcal{L}^+_\Box$ such that S5 $\vdash \Phi$ iff MEL$^{+T} \vdash \Phi'$.*

What MEL, GPL and MEL^+, GPL^+ are good for

- A belief base in GPL typically contains what an observer $\mathcal{A}$ knows about the knowledge of an agent $\mathcal{B}$.
- In GPL^+, agent $\mathcal{A}$ is allowed to add what is known about the real world in the form of standard propositions.
- GPL^+ suggests that the epistemic state of the observer is $(F, \mathcal{E})$ whereby F is what the observer knows about the world and $\mathcal{E}$ is what he knows about the epistemic state of the other agent.
  - If $\mathcal{A}$ considers that $\mathcal{B}$'s beliefs are always correct, the former can assume axiom T is valid, thus he reasons in GPL^+T to strengthen his own knowledge of the real world.
  - Alternatively, $\mathcal{A}$ may mistrust $\mathcal{B}$ and may wish to take advantage of knowing wrong beliefs of $\mathcal{A}$; , thus he reasons in GPL^+

Conclusion

- Usual semantics of epistemic logics based on accessibility relations are not very natural for reasoning about incomplete information with an external point of view on agents
- Despite proximity of languages with KD45 and S5, the fragment GPL^{+} (resp. GPL^{+T}) has simplified semantics that:
  - are more intuitive than equivalence relations.
  - are closer to the setting of uncertainty theories
- S5, with equivalence relations semantics, is more naturally the logic of rough sets (studied by Luis. F. with E. Orlowska)
- MEL, GPL are closer to logic programming, than to the epistemic logic introspective tradition (e.g. GPL captures Answer-set Programming - DP Schockaert, KR2012)