Unification in epistemic logics

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① Classical logic (CL)

$\phi := x \mid \bot \mid (\phi \vee \psi) \mid (\phi \wedge \psi) \mid (\phi \to \psi)$.

$\phi(x_1, \dots, x_n)$ is unifiable iff there exists formulas $\psi_1, \dots, \psi_n$ such that $\models_{CL} \phi(\psi_1, \dots, \psi_n)$.

<u>Fact 1</u>: ϕ is unifiable iff ϕ is classically satisfiable.

Thus, unifiability is NP-complete.

Let ϕ be a formula and σ be a substitution, i.e. $\sigma : x_i \mapsto \sigma(x_i) = \psi_i$.

Let $\tau : x_i \mapsto \tau(x_i) = (\phi \wedge x_i) \vee (\neg\phi \wedge \sigma(x_i))$ (Löwenheim Formula).

<u>Fact 2</u>: For all formulas ψ, $\models_{CL} \phi \to (\tau(\psi) \leftrightarrow \psi)$ and $\models_{CL} \neg\phi \to (\tau(\psi) \leftrightarrow \sigma(\psi))$.

<u>Fact 3</u>: If σ is a unifier of ϕ then τ is a unifier of ϕ and τ is more general than σ, i.e. if $\models_{CL} \sigma(\phi)$ then $\models_{CL} \tau(\phi)$ and there exists a substitution μ such that for all variables x, $\models_{CL} \sigma(x) \leftrightarrow \mu(\tau(x))$;

Thus, unification is unitary. τ is more general than all unifiers of ϕ.

Example: Let $\phi(x,y,z) = (x \vee \neg y) \wedge \neg z$. Then $\sigma : \left| \begin{array}{l} x \mapsto y \\ y \mapsto y \\ z \mapsto \bot \end{array} \right|$ is a unifier of ϕ.

(2) Intuitionistic logic (IL)

Fact 4: ϕ is unifiable iff ϕ is classically satisfiable.

Thus, unifiability is NP-complete.

Ghilardi (1999) has proved that every unifiable formula ϕ possesses a finite minimal complete set of unifiers, i.e. there exists $n \geq 1$ and there exists unifiers $\sigma_1, \ldots, \sigma_n$ of ϕ such that for all unifiers σ of ϕ, some σ_i is more general than σ. Thus, unification is finitary.

Remark that unification cannot be unitary. To show this, consider the formula $\phi(x) = x \vee \neg x$. It possesses two unifiers: $\sigma_\perp(x) = \perp$ and $\sigma_\top(x) = \top$. By the disjunction property of intuitionistic logic, each unifier of ϕ is equivalent to $\sigma_\perp$ or $\sigma_\top$.

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(3) S5, KD45, K45

$$\phi ::= x \mid \perp \mid (\phi \vee \psi) \mid (\phi \wedge \psi) \mid (\phi \to \psi) \mid \Box\phi \mid \Diamond\phi .$$

Fact 5: $\phi(x_1, \ldots, x_n)$ is S5-unifiable iff there exists $\psi_1, \ldots, \psi_n \in \{\perp, \top\}$ such that $\models_{S5} \phi(\psi_1, \ldots, \psi_n)$.

Thus, S5-unifiability is NP-complete.

Fact 6 : $\phi(x_1, \dots, x_n)$ is KD45-unifiable iff there exists $\psi_1, \dots, \psi_n \in \{\bot, \top\}$ such that $\models_{KD45} \phi(\psi_1, \dots, \psi_n)$.

Thus, KD45-unifiability is NP-complete.

Fact 7 : $\phi(x_1, \dots, x_n)$ is K45-unifiable iff there exists $\psi_1, \dots, \psi_n \in \{\bot, \top, \Box\bot, \Diamond\top\}$ such that $\models_{K45} \phi(\psi_1, \dots, \psi_n)$.

Thus, K45-unifiability is NP-complete.

Let ϕ be a formula and σ be a substitution.

Let $\tau : x \longmapsto \tau(x) = (\Box\phi \wedge x) \vee (\Diamond\neg\phi \wedge \sigma(x))$.

Fact 8 : For all formulas ψ, $\models_{S5} \Box\phi \to (\tau(\psi) \leftrightarrow \psi)$ and $\models_{S5} \Diamond\neg\phi \to (\tau(\psi) \leftrightarrow \sigma(\psi))$.

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Fact 9 : If σ is an S5-unifier of ϕ then τ is an S5-unifier of ϕ and τ is more S5-general than σ; τ is more general that all unifiers of ϕ.

Thus, S5-unification is unitary.

Example : Let $\phi(x) = \Box x \vee \Box\neg x$. Then, $\sigma(x) = \Box y$ is an S5-unifier of ϕ. Let $\tau : x \mapsto \tau(x) = (\Box\phi \wedge x) \vee (\Diamond\neg\phi \wedge \sigma(x))$. Then, we have : $\tau(x) = \Box x \vee (\Diamond x \wedge \Box y)$ is a S5-(most general unifier) of ϕ.

As for unification in KD4S (and K4S), we know it is either unitary or nullary. This comes from the fact that for all formulas ϕ and for all substitutions σ, τ, if σ and τ are KD4S-unifiers (K4S-unifiers) of ϕ then the substitution $\mu : x \mapsto \mu(x) = (\Box y \wedge \sigma(x)) \vee (\Diamond \neg y \wedge \tau(x))$ (where y is a new variable) is also a KD4S-unifier of ϕ and is more KD4S-general (more K4S-general) that σ and τ.

4) $S5_n$, $KD4S_n$, $K4S_n$ $(n \geq 2)$

$$\phi ::= x \mid \bot \mid (\phi \vee \psi) \mid (\phi \wedge \psi) \mid (\phi \to \psi) \mid \Box_i \phi \mid \Diamond_i \phi \qquad 1 \leq i \leq n.$$

$S5_n$-unification and $KD4S_n$-unification are still NP-complete. This follows from the fact that $\Diamond_i \top$ is valid in these logics. As for $K4S_n$-unification, its computability is unknown.

Concerning the unification types of $S5_n$ and $KD4S_n$, they are not known. As for $K4S_n$, it is nullary. To show this, it suffices to consider the formula $\phi(x) = (x \to \Box_1 \Box_2 x)$ and to demonstrate it does not possess a minimal complete set of unifiers.

References

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