Logical Foundations of Well-Founded Semantics

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Outline

1 Introduction
   - Logical foundations of Logic Programming

2 Contributions
   - Classification of $HT^2$ frames
   - Axiomatisation of $HT^2$
   - 6-valued matrix
   - Capturing partial stable models
   - Strong equivalence

3 Conclusions
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Fixing logical foundations for LP

- LP definitions rely on:
  - syntax transformations ("reduct") + fixpoint constructions
  - Example: “$M$ is the minimal model of $\Pi^M$”

- A logical style definition:
  - get minimal models inside some (monotonic) logic.

- Logically equivalent programs $\Rightarrow$ same minimal models.

- Full logical interpretation of connectives.
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Stable models successfully identified

- (Monotonic) intermediate logic of *here-and-there* (HT) (a.k.a. Gödel’s 3-valued logic)
  - Classical ⊆ HT ⊆ Intuitionistic
  - Pearce’s *Equilibrium Logic*: minimal HT models
    Equilibrium models = stable models [Pearce 97]
  - $\Pi_1$ and $\Pi_2$ are *strongly equivalent* iff they are
    HT-equivalent [Lifschitz, Pearce & Valverde 01]
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Possible reasons:

- No logic could be identified as deductive basis for *WFS*. Intuitionistic is too strong. Example: signature \{A, B\}

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A first solution: $HT^2$ frames

- $HT^2$ [Cabalar 01]: each $HT$ world has a primed “version”

Relation $\leq$

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![Diagram](ht2_frames.png)

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![Diagram showing $HT^2$ frames]

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Relation $\leq$ implication

Relation $R$ negation

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In this work . . .

1. **[Došen 86]** framework $N$
   - Negation as a modal operator.
   - Weaker than intuitionistic and Johansson minimal logic.
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Došen logic $N$

- Inference rules: modus ponens plus
  \[
  \alpha \rightarrow \beta \quad \frac{}{\neg \beta \rightarrow \neg \alpha}
  \]

- Axioms: positive logic plus
  \[\neg \alpha \land \neg \beta \rightarrow \neg (\alpha \lor \beta)\]

- Models: an extra accessibility relation $R$ is used for negation
**Definition (N model)**

is a quadruple $\mathcal{M} = \langle W, \leq, R, V \rangle$ such that:

1. $W$ non-empty set of worlds
2. $\leq$ partial ordering among worlds
3. $R$ accessibility relation s.t. $(\leq R) \subseteq (R \leq^{-1})$
4. $V$ valuation function $At \times W \rightarrow \{0, 1\}$ satisfying:
   - $V(\varphi \rightarrow \psi, w) = 1 \quad \& \quad w \leq w' \Rightarrow V(\varphi, w') = 1$
   - $V(\neg \varphi, w) = 1 \quad \text{iff} \quad \forall w' \text{ such that } wRw' \quad V(\varphi, w') = 0$
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Routley variant $N^*$

- **Axioms:** $N$ plus
  
  $\neg (\alpha \rightarrow \alpha) \rightarrow \beta$
  
  $\neg (\alpha \land \beta) \rightarrow \neg \alpha \lor \neg \beta$

- Intuitionistic negation ‘$-$’ is definable in $N^*$ as:
  
  $-\alpha := \alpha \rightarrow \neg (\rho_0 \rightarrow \rho_0)$.

**Definition ($N^*$ model)**

is an $N$ model satisfying for all $x$, there exists the $\leq$-greatest $x^*$ $R$-accessible from $x$
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Routley style semantics

\[ x \models \neg \varphi \iff x^* \not\models \varphi \]

Definition (Routley frame)

is a triple \( \langle W, \leq, \ast \rangle \) with \( W \) and \( \leq \) as before and \( \ast : W \to W \) is such that \( x \leq y \) iff \( y^* \leq x^* \)

- Completeness: obtained via canonical model
$HT^2$ as an $N^*$ frame

An $HT^2$ frame corresponds to a $N^*$ frame with $W = \{h, h', t, t'\}$ and

where "higher" means $\leq$-greater
and the arrow represents the action of $*$
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The axioms of $HT^2$

Let $HT^*$ extend $N^*$ by adding rule $\frac{\alpha \lor (\beta \land \neg \beta)}{\alpha}$ and:

A1. $\neg \alpha \lor \neg \alpha$

A2. $\neg \alpha \lor (\alpha \rightarrow (\beta \lor (\beta \rightarrow (\gamma \lor \neg \gamma))))$

A3. $\land_{i=0}^{2} ((\alpha_i \rightarrow \lor_{j \neq i} \alpha_j) \rightarrow \lor_{j \neq i} \alpha_j) \rightarrow \lor_{i=0}^{2} \alpha_i$

A4. $\alpha \rightarrow \neg \neg \alpha$

A5. $\alpha \land \neg \alpha \rightarrow \neg \beta \lor \neg \neg \beta$

A6. $\neg \alpha \land \neg (\alpha \rightarrow \beta) \rightarrow \neg \neg \alpha$

A7. $\neg \neg \alpha \lor \neg \neg \beta \lor \neg (\alpha \rightarrow \beta) \lor \neg \neg (\alpha \rightarrow \beta)$

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A1 (Weak excluded middle for ‘$-$’)

strongly directed frame
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A2 Bounds the depth to 2 worlds
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Let $HT^*$ extend $N^*$ by adding rule $\frac{\alpha \lor (\beta \land \neg \beta)}{\alpha}$ and:

A1. $-\alpha \lor \neg \neg \alpha$
A2. $-\alpha \lor (\alpha \rightarrow (\beta \lor (\beta \rightarrow (\gamma \lor \neg \gamma))))$
A3. $\exists_i \land \lnot (\exists_i \rightarrow \lor j \neq i \alpha_j) \rightarrow \lor j \neq i \alpha_j) \rightarrow \lor i = 0 \alpha_i$
A4. $\alpha \rightarrow \neg \neg \alpha$
A5. $\alpha \land \neg \alpha \rightarrow \neg \beta \lor \neg \neg \beta$
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A3 Bounds the branching to 2 worlds
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A4-A8 Fix negation $\neg$
Main result

Theorem

$HT^* = HT^2$.

Proof sketch.

Soundness easy to check using $HT^2$ semantics. Completeness relies on canonical model method and the corresp. of $HT^2$ frames as $N^*$ frames.
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HT = Gödel’s 3-valued

\[\begin{array}{ccc}
0 & \rightarrow & p \\
\downarrow & & \downarrow \\
1 & \rightarrow & p \\
\downarrow & & \downarrow \\
2 & \rightarrow & p \\
\end{array}\]

...and the tables are derived from frames.
\( HT^2 \) becomes 6-valued

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Let $H, H', T, T'$ denote sets of atoms verified at $h, h', t, t'$.

Represent a model as a pair $\langle H, T \rangle$, where $H = (H, H')$ and $T = (T, T')$.

Define the ordering $H_1 \leq H_2$ as $H_1 \subseteq H_2$ and $H'_1 \subseteq H'_2$.

Extend this to an order among models, $\preceq$, as follows:

$\langle H_1, T_1 \rangle \preceq \langle H_2, T_2 \rangle$ if: (i) $T_1 = T_2$; (ii) $H_1 \leq H_2$.

$\langle H, T \rangle$ is said to be total if $H = T$.

Definition (Partial equilibrium model)

A model $M$ of theory $\Pi$ is a partial equilibrium model of $\Pi$ if it is total and $\preceq$-minimal.
Partial equilibrium models

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A model $\mathcal{M}$ of theory $\Pi$ is a partial equilibrium model of $\Pi$ if it is total and $\preceq$-minimal.
Partial equilibrium models

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**Theorem**

For a normal or disjunctive logic program $\Pi$, $\langle T, T \rangle$ is a partial equilibrium model of $\Pi$ iff $T$ is a partial stable model of $\Pi$. 
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Outline

1. Introduction
   - Logical foundations of Logic Programming

2. Contributions
   - Classification of $HT^2$ frames
   - Axiomatisation of $HT^2$
   - 6-valued matrix
   - Capturing partial stable models
   - Strong equivalence

3. Conclusions
Partial Equilibrium Logic and Strong equivalence

Definition (Partial Equilibrium Logic (PEL))

Partial Equilibrium Logic (PEL) is characterised by truth in all partial equilibrium models.

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Two theories $\Pi_1, \Pi_2$ are said to be strongly equivalent if for any set of formulas $\Gamma$, $\Pi_1 \cup \Gamma$ and $\Pi_2 \cup \Gamma$ have the same partial equilibrium models.
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Partial Equilibrium Logic and Strong equivalence

Theorem

*Two theories* $\Pi_1, \Pi_2$ *are strongly equivalent iff they are equivalent in HT* $^2$.

Theorem (ICLP’06)

*If $\Pi_1, \Pi_2$ are not HT$^2$-equivalent, there is a $\Gamma$ such that $\Pi_1 \cup \Gamma$ and $\Pi_2 \cup \Gamma$ have different well-founded models.*
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1. $HT^2$ frames belong to Routley variant of Došen frames. Is $HT^2$ the strongest deduct. base for WFS in this family?
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Recent work

- general properties of PEL inference
- complexity
- program transformations
- programs with nested expressions
- tableaux proof system
- extensions of PEL with strong negation
- splitting theorem for theories under PEL
- reduction of $HT^2$ to $HT$
Further reading

