Logical Foundations of Well-Founded Semantics

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Outline

1. Introduction
   - Logical foundations of Logic Programming
2. Contributions
   - Classification of HT^2 frames
   - Axiomatisation of HT^2
   - 6-valued matrix
   - Capturing partial stable models
   - Strong equivalence
3. Conclusions

Outline

Fixing logical foundations for LP

- LP definitions rely on:
  syntax transformations ("*reduct*") + fixpoint constructions
  Example: "M is the minimal model of Π^M"
- A logical style definition:
  get minimal models inside some (monotonic) logic.
- Logically equivalent programs $\Rightarrow$ same minimal models.
- Full logical interpretation of connectives.

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Stable models successfully identified

- (Monotonic) intermediate logic of *here-and-there* (*HT*)
  (a.k.a. Gödel's 3-valued logic)

  Classical $\subseteq$ *HT* $\subseteq$ Intuitionistic

- Pearce's *Equilibrium Logic*: minimal *HT* models
  Equilibrium models = stable models [Pearce 97]

- Π_1 and Π_2 are *strongly equivalent* iff they are *HT*-equivalent [Lifschitz, Pearce & Valverde 01]

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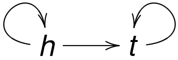

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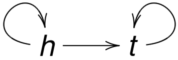

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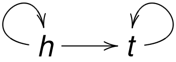

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Logical foundation for *WFS* was missing

Possible reasons:

- No logic could be identified as deductive basis for WFS.
  Intuitionistic is too strong. Example: signature $\{A, B\}$

Program	WFS
$\neg A \rightarrow A$	A undefined, B

- Good algorithmic properties, but poor model-based defs.
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A first solution: HT^2 frames

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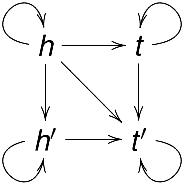

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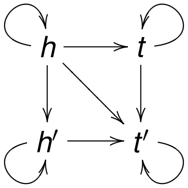

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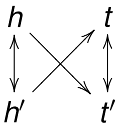

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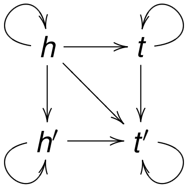

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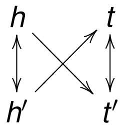

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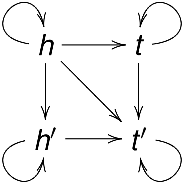

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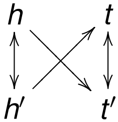

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1. [Došen 86] framework N
   - Negation as a modal operator.
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Outline

Došen logic *N*

- Inference rules: modus ponens plus $\frac{\alpha \rightarrow \beta}{\neg\beta \rightarrow \neg\alpha}$
- Axioms: positive logic plus $\neg\alpha \wedge \neg\beta \rightarrow \neg(\alpha \vee \beta)$
- Models: an extra accesibility relation *R* is used for negation

Došen logic N

Definition (N model)

is a quadruple $\mathcal{M} = \langle W, \leq, R, V \rangle$ such that:

1. W non-empty set of worlds
2. $\leq$ partial ordering among worlds
3. R accessibility relation s.t. $(\leq R) \subseteq (R \leq^{-1})$
4. V valuation function $At \times W \longrightarrow \{0,1\}$ satisfying:
   $V(p,w) = 1 \ \ \& \ \ w \leq w' \ \ \Rightarrow \ \ V(p,w') = 1$

- $V(\varphi \rightarrow \psi, w) = 1$ iff $\forall w'$ such that $w \leq w'$, $V(\varphi, w') = 0$ or $V(\psi, w') = 1$.
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Routley variant N^*

- Axioms: N plus
  $\neg(\alpha \to \alpha) \to \beta$
  $\neg(\alpha \wedge \beta) \to \neg\alpha \vee \neg\beta$

- Intuitionistic negation '$-$' is definable in N^* as:
  $-\alpha \ := \ \alpha \to \neg(p_0 \to p_0)$.

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Routley style semantics

- $x \models \neg\varphi$ iff $x^* \not\models \varphi$

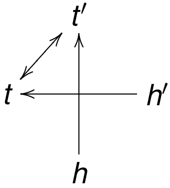
Definition (Routley frame)

is a triple $\langle W, \leq, * \rangle$ with W and $\leq$ as before and $* : W \rightarrow W$ is such that $x \leq y$ iff $y^* \leq x^*$

- Completeness: obtained via canonical model

HT^2 as an N^* frame

- An HT^2 frame corresponds to a N^* frame with $W = \{h, h', t, t'\}$ and



where "higher" means $\leq$-greater
and the arrow represents the action of $*$

Outline

The axioms of HT^2

Let HT^* extend N^* by adding rule $\frac{\alpha \vee (\beta \wedge \neg \beta)}{\alpha}$ and:

A1. $-\alpha \vee - - \alpha$

A2. $-\alpha \vee (\alpha \rightarrow (\beta \vee (\beta \rightarrow (\gamma \vee -\gamma))))$

A3. $\bigwedge_{i=0}^{2}((\alpha_i \rightarrow \bigvee_{j \neq i} \alpha_j) \rightarrow \bigvee_{j \neq i} \alpha_j) \rightarrow \bigvee_{i=0}^{2} \alpha_i$

A4. $\alpha \rightarrow \neg\neg\alpha$

A5. $\alpha \wedge \neg\alpha \rightarrow \neg\beta \vee \neg\neg\beta$

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A1 (Weak excluded middle for '$-$') strongly directed frame

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A2 Bounds the depth to 2 worlds

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A3 Bounds the branching to 2 worlds

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A4-A8 Fix negation $\neg$

Main result

Theorem

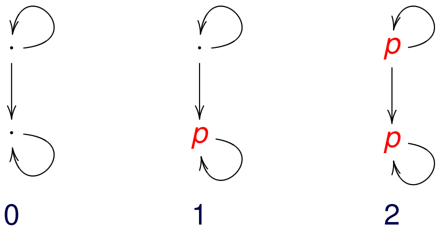
$HT^* = HT^2$.

Proof sketch.

Soundness easy to check using HT^2 semantics.
Completeness relies on canonical model method and the corresp. of HT^2 frames as N^* frames. □

Outline

HT = Gödel's 3-valued



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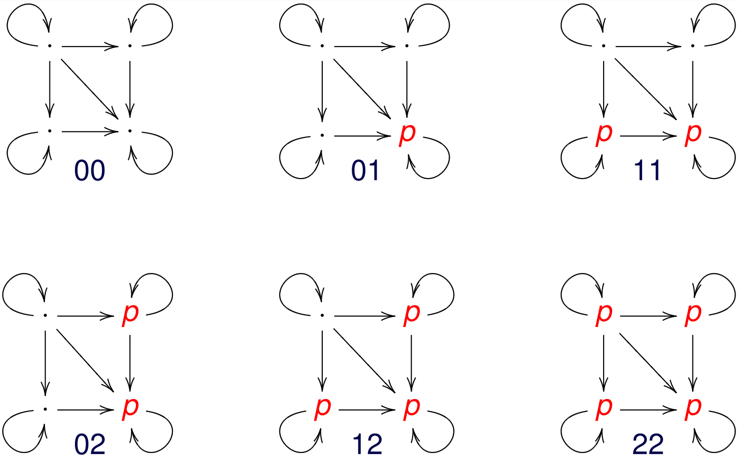
HT^2 becomes 6-valued



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Partial equilibrium models

- Let H, H', T, T' denote sets of atoms verified at h, h', t, t'.
- Represent a model as a pair $\langle \mathbf{H}, \mathbf{T} \rangle$, where $\mathbf{H} = (H, H')$ and $\mathbf{T} = (T, T')$.
- Define the ordering $\mathbf{H}_1 \leq \mathbf{H}_2$ as $H_1 \subseteq H_2$ and $H'_1 \subseteq H'_2$.
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- $\langle \mathbf{H}, \mathbf{T} \rangle$ is said to be *total* if $\mathbf{H} = \mathbf{T}$.

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A model $\mathcal{M}$ of theory Π is a *partial equilibrium model* of Π if it is total and $\trianglelefteq$-minimal.

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- $\langle \mathbf{H}, \mathbf{T} \rangle$ is said to be *total* if $\mathbf{H} = \mathbf{T}$.

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A model $\mathcal{M}$ of theory Π is a *partial equilibrium model* of Π if it is total and $\trianglelefteq$-minimal.

Partial equilibrium models

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For a normal or disjunctive logic program Π, $\langle \mathbf{T}, \mathbf{T} \rangle$ *is a partial equilibrium model of* Π *iff* $\mathbf{T}$ *is a partial stable model of* Π.

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Outline

1. Introduction
   - Logical foundations of Logic Programming
2. Contributions
   - Classification of HT^2 frames
   - Axiomatisation of HT^2
   - 6-valued matrix
   - Capturing partial stable models
   - Strong equivalence
3. Conclusions

Partial Equilibrium Logic and Strong equivalence

Definition (Partial Equilibrium Logic (PEL))

Partial Equilibrium Logic (PEL) is characterised by truth in all partial equilibrium models.

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Two theories Π_1, Π_2 are said to be *strongly equivalent* if for any set of formulas Γ, $\Pi_1 \cup \Gamma$ and $\Pi_2 \cup \Gamma$ have the same partial equilibrium models.

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Summary

A deductive base for WFS is now identified:

1. HT^2 frames belong to Routley variant of Došen frames. Is HT^2 the strongest deduct. base for WFS in this family?
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Recent work

- general properties of PEL inference
- complexity
- program transformations
- programs with nested expressions
- tableaux proof system
- extensions of PEL with strong negation
- splitting theorem for theories under PEL
- reduction of HT^2 to HT

Further reading

- P. Cabalar, S. Odintsov, D. Pearce & A. Valverde. Analysing and Extending Well-Founded and Partial Stable Semantics using Partial Equilibrium Logic. In *Proceedings ICLP 06*, to appear.
- P. Cabalar, S. Odintsov & D. Pearce. Strong Negation in Well-Founded and Partial Stable Semantics for Logic Programs. In *Proceedings of IBERAMIA'06*, (LNCS, to appear).
- P. Cabalar, S. Odintsov, D. Pearce & A. Valverde. On the logic and computation of Partial Equilibrium Models (extended abstract). Unpublished draft available at `http://www.dc.fi.udc.es/~cabalar/lcpem.pdf`.