On the notion of f-inclusion and its logic

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Introduction

- The concept of inclusion is undoubtedly one of the most important concepts in set theory.
- There is currently no consensus on how to extend this concept. Most researchers favour two options.
  - The constructive one, in which we have

$$S(A, B) = \bigwedge_{u \in \mathcal{U}} A(u) \to B(u),$$

  - The axiomatic one, for instance the works of Sinha-Dougherty and Kitainik .

Axioms of Sinha-Dougherty

Definition

The mapping $S\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ is an *SD-measure of inclusion* if the following axioms hold for all fuzzy sets A, B and C:

- **(SD1)** $S(A,B) = 1$ if and only if $A(u) \le B(u)$ for all $u \in \mathcal{U}$.
- **(SD2)** $S(A,B) = 0$ if and only if there exists $u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0$.
- **(SD3)** Si $B(u) \le C(u)$ for all $u \in \mathcal{U}$ then $S(A,B) \le S(A,C)$.
- **(SD4)** Si $B(u) \le C(u)$ for all $u \in \mathcal{U}$ then $S(C,A) \le S(B,A)$.
- **(SD5)** If $T\colon \mathcal{U} \to \mathcal{U}$ is a bijective mapping in the universe, then $S(A,B) = S(T(A), T(B))$.
- **(SD6)** $S(A,B) = S(B^c, A^c)$.
- **(SD7)** $S(A \cup B, C) = \min\{S(A,C), S(B,C)\}$.
- **(SD8)** $S(A, B \cap C) = \min\{S(A,B), S(A,C)\}$.

f-index of inclusion

The f-index of inclusion can be defined in three steps.

Definition

The set of f-indexes of inclusion, Ω, is the set of all the increasing mappings $f\colon [0,1] \to [0,1]$ satisfying $f(x) \leq x$ for all $x \in [0,1]$.

Definition

Let A and B be fuzzy sets and consider $f \in \Omega$. We say that A *is f-included in* B (denoted $A \subseteq_f B$) if and only if $f(A(u)) \leq B(u)$ holds for all $u \in \mathcal{U}$.

Definition

Let A and B be fuzzy sets, the *f-index of inclusion* of A in B, denoted $\mathrm{Inc}(A,B)$, is defined as

$$\mathrm{Inc}(A,B) = \max\{f \in \Omega \mid A \subseteq_f B\}$$

Theorem

Given fuzzy sets A, B and C

1. $Inc(A, B) = id$ *if and only if* $A(u) \leq B(u)$ *for all* $u \in \mathcal{U}$.
2. $Inc(A, B) = \perp$ *if and only if there exists a sequence* $\{u_i\}_{i \in \mathbb{N}} \subseteq \mathcal{U}$ *such that* $A(u_i) = 1$ *for all* $i \in \mathbb{N}$ *and* $\bigwedge_{i \in \mathbb{N}} B(u_i) = 0$.
3. *If* $B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $Inc(C, A) \leq Inc(B, A)$.
4. *If* $B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $Inc(A, B) \leq Inc(A, C)$;
5. *Let* $T : \mathcal{U} \to \mathcal{U}$ *be a transformation on* $\mathcal{U}$, *then* $Inc(A, B) = Inc(T(A), T(B))$.
6. $Inc(A, B \cap C) = Inc(A, B) \wedge Inc(A, C)$.
7. $Inc(A \cup B, C) = Inc(A, C) \wedge Inc(B, C)$.

f-index of inclusion (comments)

Concerning (SD6), the following *measure of inclusion* (based on the f-index of inclusion) was introduced.

Definition

Let A and B be fuzzy sets, the *measure of inclusion* of A in B induced by the f-index of inclusion is defined by

$$M_{\mathrm{Inc}}(A, B) = 2 \int_0^1 \mathrm{Inc}(A, B)(x) dx.$$

Theorem

Let A and B be fuzzy sets, then $M_{Inc}(A, B) = M_{Inc}(B^c, A^c)$.

f-inclusion and modus ponens

The f-index of inclusion has been related to the adjoint pairs by the following result:

Theorem

Let A and B be fuzzy sets defined on a finite universe $\mathcal{U}$, then there exists an adjoint pair $(*, \to)$ such that

$$Inc(A; B)(x) = x * \left(\bigwedge_{u \in \mathcal{U}} A(u) \to B(u) \right)$$

Notice the relationship with Zadeh's compositional rule of inference.

f-inclusion and modus ponens

As a consequence, the f-index of inclusion *à la* Modus Ponens.

$$\begin{array}{lll} & A \Rightarrow B & \equiv \mathrm{Inc}(A;B) \\ & A(u) & \equiv \ \beta \\ \hline \therefore & B(u) & \geq \mathrm{Inc}(A;B)(\beta) \end{array}$$

- We obtain that the inference done in terms of the f-index of inclusion is the greatest among all the possible inferences that can be done in terms of adjoint pairs.
- Initial results point to the possible existence of Generalized Modus Ponens which satisfies of the properties stated by Baldwin and Pilsworth; always reinterpreting the symmetry in terms of Galois connections.

IŁ$_\Delta$: A logic for f-inclusion

Together with Lluís Godo and Tommaso Flaminio

Definition (Formulas of IŁ$_\Delta$)

- *FM*:
  Built from a set of variables *Var* using Łukasiewicz logic connectives $\rightarrow, \&, \overline{0}$
- *FMI*:
  Basic: for each pair of formulas $\varphi, \psi, \chi \in$ *FM*, $I_{\varphi,\psi}(\chi) \in$ *FMI*
  Complex: if $\varphi, \psi \in$ *FMI* then $\varphi \rightarrow \psi, \varphi \& \psi \in$ *FMI*

IŁ$_\Delta$: A logic for f-inclusion

Together with Lluís Godo and Tommaso Flaminio

Definition (Axioms of IŁ$_\Delta$)

- Axioms and rules of Łukasiewicz logic $+ \Delta$
- Monotonies of $I_{\varphi,\psi}$'s:
  $\Delta(\varphi \to \varphi') \to (I_{\varphi',\psi}(\chi) \to I_{\varphi,\psi}(\chi))$
  $\Delta(\psi \to \psi') \to (I_{\varphi,\psi}(\chi) \to I_{\varphi,\psi'}(\chi))$
  $\Delta(\chi \to \chi') \to (I_{\varphi,\psi}(\chi) \to I_{\varphi,\psi}(\chi'))$
- $I_{\varphi,\psi}(\chi) \to \chi$
- $I_{\varphi,\psi}(\varphi) \to \psi$
- $\Delta(I_{\varphi',\psi'}(\varphi) \to \psi) \to (I_{\varphi',\psi'}(\chi) \to I_{\varphi,\psi}(\chi))$
- $\psi \to I_{\varphi,\varphi}(\psi)$

IŁ$_\Delta$: A logic for f-inclusion

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Definition (Semantics: IMV$_\Delta$-algebras)

An IMV$_\Delta$-algebra is an MV$_\Delta$-algebra $\mathbf{A} = (A, \to, \otimes, 0, 1, \Delta)$ together with a ternary operator $I : A \times A \times A \to A$ satisfying:

1. $I(\downarrow, \uparrow, \uparrow)$
2. $I(a, b, a) \leq b$
3. $I(a, b, c) \leq c$
4. $b \leq I(a, a, b)$
5. If $I(a', b', a) \leq b$ then $I(a', b', c) \leq I(a, b, c)$

IŁ$_\Delta$: A logic for f-inclusion

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Theorem (Chain completeness of IŁ$_\Delta$)

The logic IŁ$_\Delta$ is complete wrt to linearly-ordered IMV$_\Delta$-algebras.

Proof (Sketch).

Based on the fact that all the conditions can be expressed as equations (with Δ) and the ternary operation preserves congruences.

$\square$

Future work

- Further theoretical properties of the f-index of inclusion:
  - The relation with the f-index of contradiction and the resulting opposition square.
  - The residuated structures that can be defined on the set of f-index of inclusion.
  - The induced entropy and similarity measures.
- Standard completeness of the $I Ł_\Delta$.

Previous works (selection)

- NM, MOA, and I. Perfilieva. *f-inclusion indexes between fuzzy sets*. In Proc. of IFSA-EUSFLAT, 2015
- H. Bustince, NM, MOA. *The Notion of Weak-Contradiction: Definition and Measures*. IEEE Trans. Fuzzy Systems, **23**(4) (2015), 1057–1069.
- NM, MOA. *On contradiction and inclusion using functional degrees*. Intl J. of Computational Intelligence Systems, **13**(1) (2020), 464–471.
- NM, MOA. *Functional degrees of inclusion and similarity between L-fuzzy sets*. Fuzzy Sets and Systems, **390** (2020), 1–22.
- NM, MOA. *Measures of inclusion and entropy based on the φ-index of inclusion*. Fuzzy Sets and Systems, **423** (2021), 29–54.
- NM and C. Cornelis. *Kitainik axioms do not characterize the class of inclusion measures based on contrapositive fuzzy implications*. Fuzzy Sets and Systems. Accepted (In press).
- NM, MOA. *The f-index of inclusion as optimal adjoint pair for fuzzy modus ponens*. (Submitted)

On fuzzy closure systems for fuzzy consequence operators

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Let me paint you a situation

A woman is mugged on the street, she is in the police station giving a declaration.

Statement

"It was a furious, very tall and very strong man, I remember his piercing green eyes."

What she describes

The real mugger

Thus, fuzzy logic

We need a mathematical structure capable of working with

- Uncertainty
- Vagueness
- Incompleteness
- Errors

Fuzzy logic gives a framework where working with this kind of data is within reach.

(Fuzzy) closure operators

Definition ((Classical) closure operator)

Let $(A, \leq)$ be a partially ordered set. A mapping $\mathrm{c}\colon A \to A$ is said to be a (classical) closure operator if it satisfies the following:

1. $x \leq \mathrm{c}(x)$, for all $x \in A$
2. if $x \leq y$ then $\mathrm{c}(x) \leq \mathrm{c}(y)$, for all $x, y \in A$
3. $\mathrm{c}(\mathrm{c}(x)) = \mathrm{c}(x)$, for all $x \in A$.

Definition (Closure operator)

Let (A, ρ) be a fuzzy partially ordered set. A mapping $\mathrm{c}\colon A \to A$ is said to be a (fuzzy) closure operator if it satisfies the following:

1. $\rho(x, \mathrm{c}(x)) = 1$, for all $x \in A$
2. $\rho(x, y) \leq \rho(\mathrm{c}(x), \mathrm{c}(y))$, for all $x, y \in A$
3. $\rho(\mathrm{c}(\mathrm{c}(x)), \mathrm{c}(x)) = 1$, for all $x \in A$.

Fuzzy closure systems

Intuition

We would like closure systems in the fuzzy framework to extend the meet-subsemilattice property of closure systems in the classical case.

Definition

A crisp set $\mathcal{F} \subseteq \boldsymbol{A}$ is said to be a closure system if $\bigsqcap \boldsymbol{X} \in \mathcal{F}$, for all $\boldsymbol{X} \in \boldsymbol{L}^{\mathcal{F}}$.

Theorem

There is a one-to-one connection between closure systems and closure operators.

Fuzzy closure systems

First idea

Fuzzy closure systems would be a fuzzy sets such that the membership of an element is exactly the degree to which an element is closed.

Success

The previous idea was successful and we could formalize a (somewhat technical) definition of fuzzy closure systems.

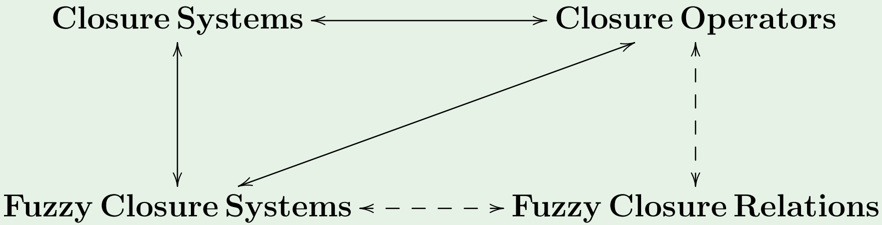
Theorem

There is a one-to-one connection between fuzzy closure systems and closure operators.

Fuzzy closure relations

Intuition

Once the closure systems have been extended to fuzzy sets, we now focus on extending the closure operators to closure relations in order to complete a diagram similar to the following one.



Fuzzy closure relations

Intuition

We would like a fuzzy closure relation $\kappa(a, b)$ to represent how close is b of being the closure of a, that is, how similar are $\mathrm{c}(a)$ and b.

Results

Again, following this idea we find a (even more technical) definition of what a fuzzy closure relation should be. This definition turned out to be equivalent to some other approaches in the literature, such as perfect fuzzy functions or extensional hulls.

Theorem

There is a one-to-one relation between fuzzy closure systems and strong fuzzy closure relations.

Pseudointents

In the classical case...

There is a particular kind of elements called pseudo-closed elements, or pseudointents, that satisfy the following. The set of attribute-implications

$$\{\boldsymbol{p} \to (\boldsymbol{p}) \mid \boldsymbol{p} \text{ is a pseudointent}\}$$

is a minimal basis of attribute-implications called Duquenne-Guigues basis or stem basis.

Work in progress

The extension of this concept to the fuzzy framework depends strongly on the fuzzy closure structures defined in the slides. The search for an appropriate definition of pseudointent and the study of minimality of the basis they define is our current project.

Conclusions and further work

Conclusions

- We have done a lot of technical work to broaden the amount of tools available for applications.

Further work

- Use these recent results to face the problem that initially generated all this line of research in the first place, that is, the fuzzy extension of pseudo-intents. (Minimal basis)
- A study on fuzzy consequence operators can be done using the fuzzy closure structures presented here.

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