MATH 109 MIDTERM 1 SOLUTIONS

Wednesday, January 28, 2004

Name: ______________________________

Numeric Student ID: ____________________

Instructor's Name: ______________________

I agree to abide by the terms of the honor code:

Signature: ___________________________

Instructions: Print your name, student ID number and instructor's name in the space provided. During the test you may not use notes, books or calculators. Read each question carefully and **show all your work**; full credit cannot be obtained without sufficient justification for your answer unless explicitly stated otherwise. Underline your final answer to each question. There are 6 questions. You have 50 minutes to do all the problems.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
Total		60

1. Determine if the following sets G with the indicated operation form a group by checking the group axioms. If not, point out which of the group axioms fail.

   (a) $G = \{1, 5\}$ with operation $*$ multiplication modulo 6.

   Solution:

   Check closure and identity and inverse with a 2x2 group table. Associativity in groups under modular multiplication follows from associativity of integers under multiplication. So G is a group.

   (b) $G = \{1, 3, 5, 7, 9\}$ with operation $*$ multiplication modulo 10.

   Solution:

   This is not a group since 5 doesn't have an inverse. You can show this by either finding the inverses of 1, 3, 7, and 9 (3 is inverse of 7, 9 is own inverse) and showing 5 is not its own inverse. Or you can argue that since 5 is not relatively prime to 10, there is no such k such that $5k = 10j + 1$ for some j.

   (c) $G = \{\text{functions } f_{a,b} = ax + b \text{ such that } a, b \in \mathbb{R},\ a \neq 0\}$, the set of linear functions of a real variable x with $*$ given by composition of functions.

   Solution:

   This is a group under composition of functions.

   $$f_{c,d} \circ f_{a,b}(x) = f_{c,d}(ax + b) = c(ax + b) + d$$

   which is again a linear function with $ac \neq 0$ if $a, c \neq 0$. The identity function is the linear function which takes $x \to x$, so pick $a = 1$ and $b = 0$. For inverses, looking at the above composition, we need to pick c, d so that $(ca)x + cb + d = x$, the identity element. So choose $c = 1/a$ and $d = -b/a$ which is well-defined since $a \neq 0$. Associativity follows since composition of functions is always associative.

   (d) $G = \mathbb{R} - \{0\}$, the set of non-zero real numbers with operation $*$ given by $a * b = a^2 b$. So for example, $2 * 3 = 2^2(3) = 12$.

   Solution:

   This fails several of the group axioms. First, there is no identity element, since for any proposed identity e,

   $$a * e = a^2 e \text{ which must equal } a \text{ if } e \text{ is the identity}$$

   So then $e = 1/a$. Pick any other real number $b \neq a$ and then $b * 1/a = b^2/a \neq b$. If there's no identity element, then there is no way to define inverses. It further fails associativity:

   $$(a * b) * c = a^2 b * c = a^4 b^2 c \neq a^2 b^2 c = a * (b * c)$$

2. Show that $G = \mathbb{Z}[i] = \{a + bi$, complex numbers such that $a, b \in \mathbb{Z}\}$ forms a group under addition of complex numbers. Then find 2 proper subgroups (i.e. not $\{e\}$ or the entire group G) of this group (that is, check that the 2 proper subsets you choose satisfy the axioms required to be a subgroup).

   Solution:

   The addition in the complex numbers is given by

$$a + bi * c + di = (a + c) = (b + d)i$$

   so all of the group axioms essentially follow from those of the integers. It's closed since the sum of any two integers is an integer, hence $(a + c)$ and $(b + d)$ are both integers. The identity element is just $0 = 0 + 0i$ and inverses are $-a - bi$ for any element $a + bi$. Associativity follows since the addition in the integers is associative. To find proper subgroups, there are many approaches. In particular,

   pick any element g and consider the subgroup generated by g, denoted $< g >$. This will always be a proper subgroup since G is not cyclic. Here are some examples of that:

   (a) The integers form a group under addition, and are contained in $\mathbb{Z}[i]$ so they form a subgroup (generated by (1)).

   (b) The subgroup of all even integers, generated by the element (2).

   (c) The subgroup of all $a + bi$ such that $a = b$, generated by the element (1+i)

   (d) The subgroup of all $a + bi$ such that a is even and b is a multiple of 3. This is an example of a proper subgroup not generated by a single element. It is generated by 2 and $3i$.

3. (a) Find the order of the following elements of S_7, the symmetric group on the set $\{1, 2, \ldots, 7\}$.

   i. $(1\ \ 2)(3\ \ 4\ \ 5)(6)(7)$

   Solution:

   The least common multiple of cycle lengths is 6 and the permutation is written as a product of disjoint cycles, so the order of the element is 6.

   ii. $(1\ \ 2)(1\ \ 3)(4\ \ 3\ \ 5\ \ 7)$

   Solution:

   This permutation is not composed of disjoint cycles. We need to rewrite it as a product of disjoint ones. Doing this we find the above is equal to

   $$(4\ 2\ 1\ 3\ 5\ 7)$$

   which has order 6.

   iii. $(1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7)(1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7)$

   Solution:

   The cycle (1 2 3 4 5 6 7) is of length 7, so this element, call it x has order 7. But the permutation above is x^2. Since 2 and 7 are relatively prime, no smaller power of x^2 is taken to the identity, so the order of x^2 is also 7. (Note $(x^2)^7 = x^{14} = e$.) One could also just rewrite the above as a product of disjoint cycles:

   $$(1\ 3\ 5\ 7\ 2\ 4\ 6)$$

   which is of order 7.

(b) Find a subgroup of order 6 in S_7.

Solution:

There are lots of ways to find such a subgroup. One is to consider the subgroup generated by a single element of order 6, like either of the ones you found in part (a) or (b) above. Then the order of the subgroup is just the number of distinct powers of the generating element, so also equal to 6. There are other subgroups of order 6 as well, like S_3, which permutes just 1,2,3. These permutations are a subset of those that permute 1,2,3,4,5,6,7. They fix 4 through 7. So this is a subgroup of order 6.

4. Prove or find a counterexample to the following statement: The elements of order ≤ 3 in a group G form a subgroup.

   Solution:

   You can prove every subgroup axiom except closure, and this is the real problem. An element of order 2 and an element of order 3 typically have a product whose order is 6. (Unless the group has order 6, in which case the product may be the identity as in D_3.) There are lots of counterexamples that you can suggest. D_n for $n > 3$ works, as does S_n for $n > 3$. Or even $\mathbb{Z}/6\mathbb{Z}$ under addition. There, 2 has order 3 and 3 has order 2, but 2+3 = 5 has order 6.

5. If G is a group such that $x^2 = e$ for all x in G, prove that G is abelian.

Solution:

There are many ways to get from $x^2 = e$ for all $x \in G$ to showing $xy = yx$ for all pairs x, y in G. The key observation is that any element is its own inverse according to this assumption. Here's one proof: For any x, y in G

$$(x * y)^2 = x * y * x * y = e \quad \text{since } x * y \in G \text{ by closure}$$

But

$$x * y * y * x = x * (y * y) * x = x * e * x = e$$

using associativity, and the $x^2 = e$ assumption. So putting these together

$$x * y * x * y = x * y * y * x$$

Cancelling x and y on the left b multiplying by inverses gives the result.

6. If G is a group with an even number of elements, show that the number of elements with order exactly equal to 2 is odd. (Hint: first show that an element of order 2 always exists for such a group.)

Solution:

To show that an element of order 2 exists, note that the order of the group is even and the identity is one of these. This leaves an odd number of non-identity elements. Now G is a group so every element has a unique inverse. Pairing these odd number of elements up, at least one element in the group must be its own inverse. That is $x * x = e$. So this x is of order 2. Note that there could never be an even number of non-identity elements which are their own inverse since this would leave an odd number of remaining non-identity elements, which all must be uniquely paired with other elements. Hence, the number of non-identity elements which are their own inverse is odd. These are the elements of order 2 in G.