An Infinitary Encoding of Temporal Equilibrium Logic

Pedro Cabalar, Martín Diéguez

Concepción Vidal

Dept. of Computer Science
University of Corunna, Spain
{cabalar,martin.dieguez,concepcion.vidalm}@udc.es

submitted 1 January 2003; revised 1 January 2003; accepted 1 January 2003

Abstract

This paper studies the relation between two recent extensions of propositional Equilibrium Logic, a well-known logical characterisation of Answer Set Programming. In particular, we show how Temporal Equilibrium Logic, which introduces modal operators as those typically handled in Linear-Time Temporal Logic (LTL), can be encoded into Infinitary Equilibrium Logic, a recent formalisation that allows the use of infinite conjunctions and disjunctions. We prove the correctness of this encoding and, as an application, we further use it to show that the semantics of the temporal logic programming formalism called TEMPLOG is subsumed by Temporal Equilibrium Logic.

1 Introduction

Many applications of the paradigm of *Answer Set Programming* (ASP) (Niemelä 1999; Marek and Truszczyński 1999; Brewka et al. 2011) involve temporal reasoning in dynamic domains. In most cases, the representation of time in ASP follows the methodology proposed in (Gelfond and Lifschitz 1993) and further developed in (Gelfond and Lifschitz 1998; Baral 2003) where transition-based action theories are translated into logic programs in a systematic way. In this methodology, time has a linear structure (normally represented as an integer variable) and dynamic laws describe, for any transition, how to derive fluent values in the resulting state from their values in the previous one and the action occurrences. This situation constitutes a perfect context for the application of temporal modal operators as those typically used in *Linear-Time Temporal Logic* (LTL) (Kamp 1968; Manna and Pnueli 1991).

The combination of ASP with LTL operators has motivated the introduction of *Temporal Equilibrium Logic* (TEL) (Cabalar and Vega 2007), a modal extension of *Equilibrium Logic* (Pearce 1996; Pearce 2006), probably, the best-known logical characterisation of ASP. Recent results about TEL (Aguado et al. 2013) have shown its adequacy as a semantic framework for temporal ASP. For instance, there exists a pair of tools (Cabalar and Diéguez 2011; Cabalar and Diéguez 2014) that allow computing temporal stable models (represented as Büchi automata). These tools can be used to check verification properties that are usual in LTL, like the typical safety, liveness and fairness conditions, but in the context of temporal ASP. Moreover, they can also be applied for planning problems that involve an indeterminate or even infinite number of steps, such as the non-existence of a plan. In the theoretical realm, the monotonic basis of TEL, the logic of Temporal Here-and-There (THT), has been partially axiomatised (Balbiani and Diéguez 2015) and its satisfiability has been shown to be a sufficient and necessary condition for strong equivalence (Cabalar and Diéguez 2014). THT and TEL satisfiability have been respectively classified as PSPACE (Cabalar and Demri 2011) and EXPSPACE (Bozzelli and Pearce 2015) complete problems.

There is, however, one foundational point that remained unclear: although the semantics for TEL obviously collapses to Equilibrium Logic for theories without modal operators, it is natural to wonder up to which point the modal extension is *reasonable*. In most practical cases, temporal formulas have an ASP "reading." For instance, a formula:

$$\Box(\neg p \rightarrow \bigcirc p) \tag{1}$$

has the informal reading of an infinite set of ASP rules of the form $\neg(\bigcirc^i p) \rightarrow (\bigcirc^{i+1} p)$ for $i \geq 0$ or, if preferred $\neg p(i) \rightarrow p(i+1)$ if we reify the temporal index as a predicate argument. Without additional information, the unique stable model of such an infinite program would make p false at even states and true at odd states – this corresponds indeed to the unique temporal equilibrium model. However, in the general case, TEL semantics for *arbitrary* temporal theories had not been actually compared to anything else. Most temporal expressions do not have a direct correspondence to ASP. To put a pair of examples, consider the formulas:

$$\Diamond p \tag{2}$$

$$\neg\Box\Diamond q \rightarrow \Diamond(q \; \mathcal{U} \; p) \tag{3}$$

While (2) is still "understandable" as an existential formula $\exists i.p(i)$, something not possible in ASP but still interpretable in Quantified Equilibrium Logic, we did not have a clear intuition whether the TEL interpretation of formulas like (3) had some other reference to compare with.

In this paper we cover this aspect in two different ways: first, we show that Kamp's translation from LTL to First Order Logic is also sound for translating TEL into Quantified Equilibrium Logic. This means that there always exists a way of resorting to first-order ASP and reifying time as an argument, as we did above with $p(i)$ or $p(i+1)$, so that modal operators are replaced by standard quantifiers. Second, we provide a second translation into *Infinitary Equilibrium Logic* (Harrison et al. 2014), a recent extension that allows the use of infinite conjunctions and disjunctions. This allows, for instance, to treat (1) as the infinite conjunction $\bigwedge_{i\geq 0}[\neg(\bigcirc^i p) \rightarrow (\bigcirc^{i+1} p)]$ or (2) as the infinite disjunction $\bigvee_{i\geq 0} \bigcirc^i p$. We prove the correctness of the infinitary encoding and, as an application, we further use it to show that the semantics of the temporal logic programming formalism called TEMPLOG is subsumed by Temporal Equilibrium Logic.

The rest of the paper is structured as follows. In the next section, we recall the basic definitions from Temporal Equilibrium Logic. Section 3 describes the encoding into Quantified Equilibrium Logic and Section 4 the encoding into Infinitary Equilibrium Logic. In Section 5 we use the latter to prove the correspondence between TEMPLOG semantics and TEL. Finally, Section 6 concludes the paper. Proofs have been included in Appendix A.

2 Temporal Equilibrium Logic

The logic of *Linear Temporal Here-and-There* (THT) is defined as follows. We start from a finite set of atoms At called the *propositional signature*. The syntax of THT is the one from propositional LTL which we recall below. A temporal formula φ is defined as:

$$\varphi ::= \bot \mid p \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \rightarrow \varphi_2 \mid \bigcirc\varphi_1 \mid \varphi_1 \, \mathcal{U} \, \varphi_2 \mid \varphi_1 \, \mathcal{R} \, \varphi_2 \mid (\varphi_1)$$

where φ_1 and φ_2 are temporal formulas in their turn and p is any atom. Negation is defined as $\neg\varphi \stackrel{\text{def}}{=} \varphi \rightarrow \bot$ whereas $\top \stackrel{\text{def}}{=} \neg\bot$. Note that '$\neg$' will stand for *default negation* in all non-monotonic formalisms described in this paper. Operators $\bigcirc$, $\mathcal{U}$ and $\mathcal{R}$ are respectively read as "next," "until," and "release." As usual, $\varphi \leftrightarrow \psi$ stands for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. Other usual temporal operators can be defined in terms of $\mathcal{U}$ and $\mathcal{R}$ as follows:

$$\Box\varphi \stackrel{\text{def}}{=} \bot \, \mathcal{R} \, \varphi \qquad \Diamond\varphi \stackrel{\text{def}}{=} \top \, \mathcal{U} \, \varphi$$

Operator $\Box$ is read "forever" and $\Diamond$ stands for "eventually" or "at some future point." We define the following notation for a finite concatenation of $\bigcirc$'s

$$\bigcirc^0\varphi \stackrel{\text{def}}{=} \varphi \qquad\qquad \bigcirc^i \varphi \stackrel{\text{def}}{=} \bigcirc(\bigcirc^{i-1}\varphi) \quad (\text{with } i \geq 1)$$

An *LTL-interpretation* is an infinite sequence of sets of atoms $H_0, H_1, \ldots$ with $H_i \subseteq At$, $i \geq 0$. Given two LTL-interpretations $\mathbf{H}$ and $\mathbf{T}$, we write $\mathbf{H} \leq \mathbf{T}$ to stand for $H_i \subseteq T_i$ for all $i \geq 0$. As usual, $\mathbf{H} < \mathbf{T}$ represents $\mathbf{H} \leq \mathbf{T}$ and $\mathbf{H} \neq \mathbf{T}$, that is, the inclusion relation holds in all states but is strict $H_j \subset T_j$ for some $j \geq 0$. A *THT-interpretation* $\mathbf{M}$ is a pair of LTL-interpretations $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$, respectively standing for *here* and *there*, such that $\mathbf{H} \leq \mathbf{T}$. An interpretation $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ is said to be *total* when $\mathbf{H} = \mathbf{T}$.

Definition 1 (*THT-Satisfaction*)
We say that an interpretation $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ satisfies a formula φ at state $k \in \mathbb{N}$, written $\mathbf{M}, k \models \varphi$, when the following recursive conditions hold:

1. $\mathbf{M}, k \models p$ iff $p \in H_k$, for any $p \in At$.
2. $\mathbf{M}, k \models \varphi \wedge \psi$ iff $\mathbf{M}, k \models \varphi$ and $\mathbf{M}, k \models \psi$.
3. $\mathbf{M}, k \models \varphi \vee \psi$ iff $\mathbf{M}, k \models \varphi$ or $\mathbf{M}, k \models \psi$.
4. $\mathbf{M}, k \models \varphi \rightarrow \psi$ iff for all $\mathbf{H}' \in \{\mathbf{H}, \mathbf{T}\}$, $\langle \mathbf{H}', \mathbf{T} \rangle, k \not\models \varphi$ or $\langle \mathbf{H}', \mathbf{T} \rangle, k \models \psi$.
5. $\mathbf{M}, k \models \bigcirc\varphi$ iff $\mathbf{M}, k+1 \models \varphi$.
6. $\mathbf{M}, k \models \varphi \, \mathcal{U} \, \psi$ iff there is $j \geq k$ s.t. $\mathbf{M}, j \models \psi$ and $\mathbf{M}, i \models \varphi$ for all $i, k \leq i < j$.
7. $\mathbf{M}, k \models \varphi \, \mathcal{R} \, \psi$ iff for all $j \geq k$ s.t. $\mathbf{M}, j \not\models \psi$, then $\mathbf{M}, i \models \varphi$ for some $i, k \leq i < j$.
8. never $\mathbf{M}, k \models \bot$. ⊠

A formula φ is THT-*valid* if $\mathbf{M}, 0 \models \varphi$ for any $\mathbf{M}$. An interpretation $\mathbf{M}$ is a THT-*model* of a theory Γ, written $\mathbf{M} \models \Gamma$, if $\mathbf{M}, 0 \models \varphi$, for all formula $\varphi \in \Gamma$. It is not difficult to see that THT-satisfaction for a total interpretation $\langle \mathbf{T}, \mathbf{T} \rangle$ collapses to LTL-satisfaction for $\mathbf{T}$. As a result:

Observation 1
$\langle \mathbf{T}, \mathbf{T} \rangle \models \Gamma$ in THT if and only if $\mathbf{T} \models \Gamma$ in LTL. ⊠

Some total models will be said to be *in equilibrium* if they satisfy the following minimality condition in their "here" component.

Definition 2 (*temporal equilibrium model*)
A total THT-interpretation $\langle \mathbf{T}, \mathbf{T} \rangle$ is a *temporal equilibrium model* of a theory Γ if $\langle \mathbf{T}, \mathbf{T} \rangle \models \Gamma$ and there is no $\mathbf{H} < \mathbf{T}$, such that $\langle \mathbf{H}, \mathbf{T} \rangle \models \Gamma$. ⊠

Since a temporal equilibrium model is a total model $\langle \mathbf{T}, \mathbf{T} \rangle$, by Observation 1, it corresponds to an LTL model $\mathbf{T}$ we will call *temporal stable model.*

Definition 3 (*temporal stable model*)
If $\langle \mathbf{T}, \mathbf{T} \rangle$ is a temporal equilibrium model of a theory Γ then $\mathbf{T}$ is called a *temporal stable model* of Γ (or TS-model, for short). ⊠

As an example, take the LTL formula (2)=$\Diamond p$. Its LTL models are those sequences of states where p is true in, at least, one timepoint. This can be captured by the Büchi automaton[1] (Büchi 1962) of Figure 1(a). In order to obtain the TS-models, take any $\mathbf{T}$ where p is true more than once, say, at situations T_i and T_j, $0 \leq i < j$. We could build a sequence $\mathbf{H}$ such that, for instance, $H_i = \emptyset \subset \{p\} = T_i$ keeping all the rest unchanged with respect to $\mathbf{T}$. Obviously, $\mathbf{H} < \mathbf{T}$ whereas $\langle \mathbf{H}, \mathbf{T} \rangle \models \Diamond p$ because there still exists a future point H_j where p is true. If, on the contrary, we take $\mathbf{T}$ such that p is true only at one situation, say $T_i = \{p\}$ and $T_j = \emptyset$ for all $j \neq i$, then the only smaller LTL-interpretation $\mathbf{H}$ would make $H_j = \emptyset$ for all $j \geq 0$. However, $\langle \mathbf{H}, \mathbf{T} \rangle \not\models \Diamond p$, and thus $\langle \mathbf{T}, \mathbf{T} \rangle$ is a temporal equilibrium model, i.e., $\mathbf{T}$ is a temporal stable model. The automaton in Figure 1(b) shows the set of TS-models of $\Diamond p$ where, as we can see, p is made true once and only once.

(a) LTL-models of $\Diamond p$ (b) TS-models of $\Diamond p$

Fig. 1. LTL-models and TS-models of $\Diamond p$.

After a similar analysis, it can be also checked that the unique TS-model of (2) corresponds to an alternate sequence of states $\emptyset$,$\{p\}$,$\emptyset$,$\{p\}$,... described by the Büchi automaton depicted in Figure 2.

Without entering into formal details, its TS-models correspond to the Büchi automaton depicted in Figure 2. Essentially, there is no evidence for p at the initial situation and this makes $\bigcirc p$ true. Then, this "blocks" the rule for $\bigcirc^2 p$ which becomes false. But then, $\bigcirc^3 p$ is derived, and so on, leading to an alternate sequence of states $\emptyset$,$\{p\}$,$\emptyset$,$\{p\}$,.... With a little more effort[2], we can also verify that the TS-models of (3) are actually the same as those for $\Diamond p$, that is, they also correspond to the automaton in Figure 1(b). An informal reading of (3) is: if we cannot prove that q occurs infinitely often ($\neg \Box \Diamond q$) then make q until p ($q \; \mathcal{U} \; p$) at some arbitrary future point. As we minimise truth, we may then assume q false at all states, and then $\Diamond(q \; \mathcal{U} \; p)$ collapses to $\Diamond(\bot \; \mathcal{U} \; p) = \Diamond(\Diamond p) = \Diamond p$.

[1] A Büchi automaton accepts any infinite sequence (or word) that visits some final state infinitely often.
[2] We can also use the tool `ABSTEM` (Cabalar and Diéguez 2014) that allows the automated computation of TS-models for arbitrary temporal theories.

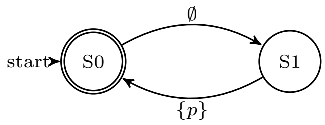

Fig. 2. TS-models of $\Box(\neg p \to \bigcirc p)$.

Note that the consequence relation induced by temporal equilibrium models is non-monotonic. In fact, when we restrict the syntax to ASP programs and the semantics to HT interpretations of the form $\langle H_0, T_0 \rangle$ we talk about (non-temporal) equilibrium models, which coincide with stable models in their most general definition (Ferraris 2005a). The result below established a more general relation to non-temporal equilibrium logic/ASP.

3 Translating TEL into Quantified Equilibrium Logic

Quantified Equilibrium Logic (Pearce and Valverde 2008) (QEL) extends Equilibrium Logic to the first-order case. As in the propositional setting, QEL defines a selection of models among those from the monotonic logic of *Quantified Here and There* (QHT).

The definition of QHT is based on a first order language denoted by $\mathcal{L} = \langle C, F, P \rangle$, where C, F and P are three disjoint sets that represent constants, functions and predicates, respectively. Given a domain D we define the sets:

- $At_D(C, P)$ stands for all atomic instances that can be formed from $\langle C \cup D, F, P \rangle$.
- $T_D(C, F)$ all ground terms that can be obtained from $\langle C \cup D, F, P \rangle$.

A *QHT-interpretation*[3] is a tuple $\mathcal{M} = \langle (D, \sigma), I_h, I_t \rangle$ such that

- $\sigma : T_D(C, F) \to D$ is a mapping from ground terms into elements of the domain satisfying that $\sigma(d) = d$ if $d \in D$
- I_h, I_t are two sets of ground atoms from $At_D(C, P)$ such that $I_h \subseteq I_t$

Given two QHT interpretations, $\mathcal{M} = \langle (D, \sigma), I_h, I_t \rangle$ and $\mathcal{M}' = \langle (D', \sigma'), I'_h, I'_t \rangle$, we say that $\mathcal{M} \leq \mathcal{M}'$ iff $D = D'$, $\sigma = \sigma'$, $T = T'$ and $H \subseteq H'$. If, additionally, $H \subset H'$ we say that the relation is strict (denoted by $\mathcal{M} < \mathcal{M}'$).

Definition 4 (*QHT semantics from (Pearce and Valverde 2008)*)
The satisfaction relation for a QHT interpretation $\mathcal{M} = \langle (D, \sigma), I_h, I_t \rangle$ is defined as follows:

- $\mathcal{M} \models \top$, $\mathcal{M} \not\models \bot$
- $\mathcal{M} \models p(\tau_1, \cdots, \tau_n)$ iff $p(\sigma(\tau_1), \cdots, \sigma(\tau_n)) \in I_h$
- $\mathcal{M} \models \tau = \tau'$ iff $\sigma(\tau) = \sigma(\tau')$.
- $\mathcal{M} \models \varphi \wedge \psi$ iff $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$
- $\mathcal{M} \models \varphi \vee \psi$ iff $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \psi$
- $\mathcal{M} \models \varphi \to \psi$ iff $\mathcal{M} \not\models \varphi$ or $\mathcal{M} \models \psi$, and $\langle (D, \sigma), I_t, I_t \rangle \models \varphi \to \psi$
- $\mathcal{M} \models \forall\, x, \varphi(x)$ iff $\mathcal{M} \models \varphi(d)$, for all $d \in D$
- $\mathcal{M} \models \exists\, x, \varphi(x)$ iff $\mathcal{M} \models \varphi(d)$, for some $d \in D$ ⊠

[3] We assume here a version of QHT taking *static* domain and *decidable* equality. Briefly, this respectively means that the domain D is common to worlds h and t and that equality is a "decidable" predicate, that is, it satisfies the excluded middle axiom $x = y \vee \neg(x = y)$.

Definition 5 (*quantified equilibrium model from (Pearce and Valverde 2008)*)
Let φ be a QHT formula. A QHT total interpretation $\mathcal{M}$ is a *first-order equilibrium model* of φ if $\mathcal{M} \models \varphi$ and there is no model $\mathcal{M}' < \mathcal{M}$ of φ. ⊠

For our purposes, it is convenient to define a particular subclass of QHT theories. We define the fragment of QHT called *monadic here-and-there with inequality*, MHT($\leq$), by syntactically restricting all predicates to monadic, excepting a binary predicate $\leq$. Moreover, we also fix the domain D to be the set of natural numbers $D = \mathbb{N}$ so that $\leq$ captures the standard ordering among them. We only consider one time constant 0 to stand for the initial situation. Given that both the domain and the interpretation of $\leq$ are fixed, interpretations will only vary for ground atoms in $At(\mathbb{N}, P)$, that is, those formed with the set of monadic predicates P and elements from $\mathbb{N}$. Then, MHT($\leq$) interpretations can be simply given by pairs $\langle \mathcal{H}, \mathcal{T} \rangle$ with $\mathcal{H} \subseteq \mathcal{T} \subseteq At(\mathbb{N}, P)$.

As usual, we write $x > y$ to stand for $\neg(x \leq y)$. We will also use the following abbreviations:

$$\begin{array}{rclcrcl}
\forall x \geq t.\, \varphi & \stackrel{\text{def}}{=} & \forall x(t \leq x \to \varphi) & \qquad & \forall x \in [t,z).\, \varphi & \stackrel{\text{def}}{=} & \forall x(t \leq x \wedge x < z \to \varphi) \\
\exists x \geq t.\, \varphi & \stackrel{\text{def}}{=} & \exists x(t \leq x \wedge \varphi) & \qquad & \exists x \in [t,z).\, \varphi & \stackrel{\text{def}}{=} & \exists x(t \leq x \wedge x < z \wedge \varphi)
\end{array}$$

Fragment MHT($\leq$) imposes exactly the same restrictions on QHT than the so-called *monadic first-order logic with inequality*, FOL($\leq$), does on classical First-Order Logic (FOL). This subclass of FOL was used by Kamp in his famous theorem (Kamp 1968) where he proved that LTL is exactly as expressive as FOL($\leq$), so that we can actually see the former as a fragment of the latter. This result was separated into two directions: proving that LTL can be translated into FOL($\leq$)and vice versa. For the first direction, Kamp defined the following translation from modal formulas into quantified first-order expressions:

Definition 6 (*Kamp's translation*)
Kamp's translation for a temporal formula φ and a timepoint $t \in \mathbb{N}$, denoted by $[\varphi]_t$, is recursively defined as follows:

$$\begin{array}{rcl}
[\bot]_t & \stackrel{\text{def}}{=} & \bot \\
[p]_t & \stackrel{\text{def}}{=} & p(t), \text{ with } p \in At. \\
[\neg\alpha]_t & \stackrel{\text{def}}{=} & \neg[\alpha]_t \\
[\alpha \wedge \beta]_t & \stackrel{\text{def}}{=} & [\alpha]_t \wedge [\beta]_t \\
[\alpha \vee \beta]_t & \stackrel{\text{def}}{=} & [\alpha]_t \vee [\beta]_t \\
[\alpha \to \beta]_t & \stackrel{\text{def}}{=} & [\alpha]_t \to [\beta]_t \\
[\bigcirc\alpha]_t & \stackrel{\text{def}}{=} & [\alpha]_{t+1} \\
[\alpha \mathbin{\mathcal{U}} \beta]_t & \stackrel{\text{def}}{=} & \exists x \geq t.\, ([\beta]_x \wedge \forall y \in [t,x).\, [\alpha]_y) \\
[\alpha \mathbin{\mathcal{R}} \beta]_t & \stackrel{\text{def}}{=} & \forall x \geq t.\, ([\beta]_x \vee \exists y \in [t,x).\, [\alpha]_y)
\end{array}$$

where $[\alpha]_{t+1}$ is an abbreviation of $\exists y \geq t.\, \neg\exists z \in [t,y).\, (t < z \wedge [\alpha]_y)$. ⊠

Note how, per each atom $p \in At$ in the temporal formula φ, we get a monadic predicate $p(x)$ in the translation.

The effect of this translation on the derived operators $\Diamond$ and $\Box$ yields the quite natural expressions:

$$[\Box\alpha]_t \equiv \forall x \geq t.\ [\alpha]_t \qquad [\Diamond\alpha]_t \equiv \exists x \geq t.\ [\alpha]_t$$

As a pair of examples, the translations of our running examples (1), (2) and (3) for $t = 0$ respectively correspond to:

$$\forall x \geq 0.\ (\neg p(x) \rightarrow p(x+1)) \tag{4}$$

$$\exists x \geq 0.\ \ p(x) \tag{5}$$

$$\forall x \geq 0.\ \Big(\forall y \geq x.\ \exists z \geq y.\ q(z) \rightarrow \exists y \geq x.\ \exists z \geq y.\ \big(p(z) \wedge \forall t \geq y.\ zq(t)\big)\Big) \tag{6}$$

Definition 7 (*THT-MHT($\leq$) interpretation correspondence*)
Given a THT interpretation $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ on a signature At, we say that the MHT($\leq$)-interpretation $\mathcal{M} = \langle \mathcal{H}, \mathcal{T} \rangle$ *corresponds* to $\mathbf{M}$ iff

- $p \in H_i$ iff $p(i) \in \mathcal{H}$, for all $i \in \mathbb{N}$.
- $p \in T_i$ iff $p(i) \in \mathcal{T}$, for all $i \in \mathbb{N}$. $\boxtimes$

We now prove that when considering this model correspondence, Kamp's translation allows us to translate a THT theory into a corresponding QHT one.

Theorem 1
Let φ be a THT formula built on a set of atoms At, $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ a THT-interpretation on At and $\mathcal{M} = \langle \mathcal{H}, \mathcal{T} \rangle$ its corresponding MHT($\leq$)-interpretation from Definition 7. It holds that $\mathbf{M}, i \models \varphi$ in THT iff $\mathcal{M} \models\ [\varphi]_i$ in MHT($\leq$). $\boxtimes$

In other words, we can consider THT as subclass of theories in the fragment MHT($\leq$) of QHT. We show next that this correspondence is still valid when we consider TS-models versus (first-order) stable models.

Theorem 2
Let $\mathbf{T}$ be a temporal interpretation, $\mathcal{T}$ its corresponding first-order interpretation and φ some temporal formula. Then, $\mathbf{T}$ is a TS-model of φ iff $\mathcal{T}$ is a stable model of $[\varphi]_0$.

4 Translating TEL into Infinitary equilibrium logic

The use of Infinitary Formulas (Scott and Tarski 1958; Karp 1964) in ASP was first proposed in (Truszczyński 2012). The syntax uses only three different connectives: $\{\}^\wedge$, which stands for an infinite conjunction, $\{\}^\vee$, which is a shorthand of an infinite disjunction and, finally, $\rightarrow$ which corresponds to the implication used in logic programming.

Definition 8 (*syntax from (Truszczyński 2012)*)
Let At be a *propositional signature* (we assume the existence of a constant $\bot$, different for all symbols in At that plays the role of *falsity*). We define the sets $\mathcal{F}_0, \mathcal{F}_1, \cdots$ by induction as follows:

1. $\mathcal{F}_0^{At} = At \cup \{\bot\}$
2. $\mathcal{F}_{i+1}^{At}$ is obtained from $\mathcal{F}_i^{At}$ by adding expressions $\mathcal{H}^\wedge$ and $\mathcal{H}^\vee$ for all subsets $\mathcal{H}$ of $\mathcal{F}_i^{At}$, and expressions $F \rightarrow G$ for all $F, G \in \mathcal{F}_i^{At}$.

The elements of $\bigcup_{i=0}^{\infty} \mathcal{F}_i^{At}$ are called (infinitary) formulas over At. ⊠

The rest of the connectives can be easily defined in terms of these ones. For instance, $G \wedge F \stackrel{\text{def}}{=} \{F, G\}^{\wedge}$, $G \vee F \stackrel{\text{def}}{=} \{F, G\}^{\vee}$, $\neg F \stackrel{\text{def}}{=} F \rightarrow \bot$, $G \leftrightarrow F \stackrel{\text{def}}{=} (F \rightarrow G) \wedge (G \rightarrow F)$ and $\top \stackrel{\text{def}}{=} \neg\bot$, as happened in THT and QHT. Several results from ASP can be extended to the case of infinitary logic. Among others, we recall here the extension of Ferraris' reduct (Ferraris 2005b) to the infinitary case (Truszczyński 2012):

Definition 9 (*from (Truszczyński 2012)*)
Let φ be a infinitary propositional formula and $\boldsymbol{I}$ an interpretation (a set of atoms). We write $\boldsymbol{I} \models \varphi$ meaning that $\boldsymbol{I}$ satisfies φ in classical logic. The *reduct* of φ relative to $\boldsymbol{I}$, denoted by $\varphi^{\boldsymbol{I}}$, is recursively defined as follows:

- If $\boldsymbol{I} \not\models \varphi$ then $\varphi^{\boldsymbol{I}} = \bot$,
- If $\boldsymbol{I} \models a$ (with a an atom) then $a^{\boldsymbol{I}} = a$,
- If $\boldsymbol{I} \models \mathcal{H}^{\wedge}$ then $(\mathcal{H}^{\wedge})^{\boldsymbol{I}} = \{\psi^{\boldsymbol{I}} \mid \psi \in \mathcal{H}^{\wedge}\}^{\wedge}$,
- If $\boldsymbol{I} \models \mathcal{H}^{\vee}$ then $(\mathcal{H}^{\vee})^{\boldsymbol{I}} = \{\psi^{\boldsymbol{I}} \mid \psi \in \mathcal{H}^{\vee}\}^{\vee}$,
- If $\boldsymbol{I} \models \varphi \rightarrow \psi$ then $(\varphi \rightarrow \psi)^{\boldsymbol{I}} = \varphi^{\boldsymbol{I}} \rightarrow \psi^{\boldsymbol{I}}$ ⊠

Broadly speaking, the reduct $\varphi^{\boldsymbol{I}}$ can be alternatively defined as the formula obtained from φ by replacing by $\bot$ every outermost subformula not satisfied by $\boldsymbol{I}$.

Definition 10 (*from (Truszczyński 2012)*)
An interpretation $\boldsymbol{I}$ is an *answer set* of a formula φ if $\boldsymbol{I}$ is the minimal model of $\varphi^{\boldsymbol{I}}$. ⊠

As an example, take the infinite disjunction φ:

$$\varphi \quad \stackrel{\text{def}}{=} \quad \{p(i) \mid i \geq 0\}^{\vee} \tag{7}$$

where we have an atom $p(j)$ in the signature for each $j \in \mathbb{N}$. Taking any interpretation $\boldsymbol{I}$ with more than one atom true, say $p(i), p(j) \in \boldsymbol{I}$ with $i < j$, the reduct $\varphi^{\boldsymbol{I}}$ would be a disjunction with all true atoms in $\boldsymbol{I}$ and, obviously, $\boldsymbol{I}$ would not be a minimal model for that disjunction (it suffices with making just one atom true). Therefore, it is clear that the answer sets of φ are all the singletons $\{p_i\}$ for any $i \geq 0$.

An alternative definition of answer sets for infinitary formulas was provided in (Harrison et al. 2014) where the authors defined infinitary versions for the logic of Here-and-There and for Equilibrium Logic. We define an *HT-interpretation* $M = \langle H, T \rangle$ on a signature At as a pair of sets of atoms $H \subseteq T \subseteq At$. As happens in the finitary case, when $H = T$ we say that M is *total*.

Definition 11 (*from (Harrison et al. 2014)*)
The HT satisfaction of formulas for infinitary formulas is defined as follows:

- $M \models p$ if $p \in H$, with $p \in At$.
- $M \models \mathcal{H}^{\wedge}$ if for every $F \in \mathcal{H}$, $M \models F$.
- $M \models \mathcal{H}^{\vee}$ if there exists $F \in \mathcal{H}$ such that $M \models F$.
- $M \models F \rightarrow G$ if either $M \not\models F$ or $M \models G$ and $\langle T, T \rangle \models F \rightarrow G$. ⊠

Definition 12 (*from (Harrison et al. 2014)*)
An HT-interpretation $\langle H, T \rangle$ is an equilibrium model of an infinitary formula $\mathcal{F}$ if $H = T$ and there is no $H' \subset T$ such that $\langle H', T \rangle \models \mathcal{F}$. $\boxtimes$

For our purposes, given a propositional signature At, our infinitary formulas will use the expanded infinite signature:

$$At^\infty \stackrel{\text{def}}{=} \{\bigcirc^i p \mid p \in At \text{ and } i \in \mathbb{N}\}$$

where we can read '$\bigcirc^i p$' altogether as an atom name. The use of infinitary operators allows an increase of expressive power with respect to LTL. For instance, the conjunction of the formulas:

$$\{\bigcirc^i p \mid i \div 2 = 0\}^\wedge \qquad \{\bigcirc^i p \vee \neg \bigcirc^i p \mid i \div 2 \neq 0\}^\wedge$$

yields to answer sets where p is true at all even states and varies freely in odd states, something that is well-known to be non-representable[4] in LTL. However, we can regard TEL as a fragment of infinitary equilibrium logic, as shown in the following translation:

Definition 13
The translation of φ into infinitary HT (HT$^\infty$) up to level $k \geq 0$, written $\langle \varphi \rangle_k$, is recursively defined as follows:

- $\langle \bot \rangle_k \stackrel{\text{def}}{=} \emptyset^\vee$
- $\langle p \rangle_k \stackrel{\text{def}}{=} \bigcirc^k p$, with $p \in At$.
- $\langle \bigcirc \varphi \rangle_k \stackrel{\text{def}}{=} \langle \varphi \rangle_{k+1}$
- $\langle \varphi \wedge \psi \rangle_k \stackrel{\text{def}}{=} \{\langle \varphi \rangle_k, \langle \psi \rangle_k\}^\wedge$
- $\langle \varphi \vee \psi \rangle_k \stackrel{\text{def}}{=} \{\langle \varphi \rangle_k, \langle \psi \rangle_k\}^\vee$
- $\langle \varphi \rightarrow \psi \rangle_k \stackrel{\text{def}}{=} \langle \varphi \rangle_k \rightarrow \langle \psi \rangle_k$
- $\langle \varphi \; \mathcal{U} \; \psi \rangle_k \stackrel{\text{def}}{=} \{\{\langle \psi \rangle_i, \langle \varphi \rangle_j \mid k \leq j < i\}^\wedge \mid k \leq i\}^\vee$
- $\langle \varphi \; \mathcal{R} \; \psi \rangle_k \stackrel{\text{def}}{=} \{\{\langle \psi \rangle_i, \langle \varphi \rangle_j \mid k \leq j < i\}^\vee \mid k \leq i\}^\wedge$ $\boxtimes$

It is easy to see that the derived operators $\Box$ and $\Diamond$ are then translated as follows:

$$\langle \Diamond \varphi \rangle_k = \{\langle \varphi \rangle_i \mid k \leq i\}^\vee \qquad \langle \Box \varphi \rangle_k = \{\langle \varphi \rangle_i \mid k \leq i\}^\wedge$$

As an example, $\langle \Diamond p \rangle_0 = \{\bigcirc^i p \mid i \geq 0\}^\vee$ which is a simple rewriting of (7). The translations for our running examples $\langle (2) \rangle_0$ and $\langle (3) \rangle_0$ respectively correspond to:

$$\{\neg \bigcirc^i p \rightarrow \bigcirc^{i+1} p \mid i \geq 0\}^\wedge$$

$$\{\{\{\bigcirc^k q \mid j \leq k\}^\vee \mid i \leq j\}^\wedge \rightarrow \{\{\bigcirc^k p, \bigcirc^h q \mid j \leq h < k\}^\wedge \mid i \leq j \leq k\}^\vee \mid i \geq 0\}^\wedge$$

We define now how a THT model is translated into HT$^\infty$ in the sense of (Harrison et al. 2014) and then we will prove that, there exists a one-to-one correspondence between the set of THT models of a formula φ and the HT$^\infty$ models of $\langle \varphi \rangle_0$.

[4] It is still an open question whether problems like this are representable in TEL or not.

Definition 14 (*HT$^\infty$-THT interpretation correspondence*)
Let $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ be a THT interpretation. We define its corresponding HT interpretation $M^\infty = \langle H^\infty, T^\infty \rangle$ as:

$$H^\infty \stackrel{\text{def}}{=} \bigcup_{i \geq 0} \{ \bigcirc^i p \mid p \in H_i \} \qquad\qquad T^\infty \stackrel{\text{def}}{=} \bigcup_{i \geq 0} \{ \bigcirc^i p \mid p \in T_i \} \qquad \boxtimes$$

Theorem 3
Let φ be a temporal formula formula, $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ a THT interpretation and $M^\infty = \langle H^\infty, T^\infty \rangle$ its corresponding HT$^\infty$ interpretation. For all $i \in \mathbb{N}$, it holds that:

(i) $\mathbf{M}, i \models \varphi$ if and only if $M^\infty \models \langle \varphi \rangle_i$.
(ii) $\mathbf{M}$ is a temporal equilibrium model of φ if and only if M^∞ is an (infinitary) equilibrium model of $\langle \varphi \rangle_0$. $\boxtimes$

5 Relation between TEL and TEMPLOG

Although the introduction of temporal operators for ASP is relatively new, the extension of Prolog with modal operators is a well-developed field that was extensively studied in the past (Fariñas del Cerro 1986; Bieber et al. 1988). In particular, there exist several proposals for extending Prolog with LTL operators (Fujita et al. 1986; Gabbay 1987; Moszkowski 1986; Orgun and Wadge 1992) being, perhaps, a prominent case the formalism called TEMPLOG (Baudinet 1992) since it provides a logical semantics in terms of a least LTL model, in the spirit of the well-known least Herbrand model for positive logic programs (van Emden and Kowalski 1976) – note that semantics for default negation were still in their early steps at that moment.

In this section we will show that TEMPLOG is actually subsumed by TEL, that is the latter can be used as a generalisation of the former for an arbitrary temporal syntax that includes default negation. The syntax of TEMPLOG is defined by the following grammar:

Non-empty Body:	B	$::= P \parallel B_1, B_2 \parallel \bigcirc B \parallel \Diamond B$
Body:	D	$::= \epsilon \parallel B$
Initial clause:	IC	$::= N \leftarrow D \parallel \Box N \leftarrow D$
Permanent clause:	PC	$::= \Box(N \leftarrow D)$
Goal clause:	G	$::= \leftarrow D$

where P stands for an atom, N for a next-atom (that is, a formula of the form $\bigcirc^i P$ for some $i > 0$) and ϵ denotes the empty expression. A TEMPLOG program is a set of *temporal clauses* that, as we can see, involve implications that can be of three types: *initial* clauses are only applicable at the initial state; *permanent* clauses are preceded by a $\Box$ operator and are always applicable; and *goal* clauses have an empty head and play the role of constraints. In all cases, both the antecedent and the consequent of each implication can be enhanced by a selective use of temporal operators.

Any TEMPLOG program can be equivalently translated into a (possibly infinite) positive program Π^* containing a set of clauses in which the only occurrences of temporal operators are in the form of next-atoms $\bigcirc^i P$. For instance, a clause with a body like $\Diamond\,(p, \bigcirc\Diamond q)$ can be replaced by the set of clauses with all bodies of the form $\bigcirc^i\,\left(p, \bigcirc^{j+1} q\right)$

for any i, j ranging on $\mathbb{N}$. In (Baudinet 1992) the $\bigcirc$ operator was further pushed inside the expressions until it only formed temporal atoms like $\bigcirc^i P$. Each clause generated in this way was called a *Temporal Ground Instance* (TGI) of the original general clause. We rephrase below the definition of TGI's in a more formal way:

Definition 15 (*Temporal Ground Instance of a body, TGI*)
Let B be a temporal body and $i \geq 0$. Then, $TGI(B, i)$ is a (possibly infinite) set of bodies recursively defined as follows:

- $TGI(\epsilon, i) \stackrel{\text{def}}{=} \emptyset$
- $TGI(P, i) \stackrel{\text{def}}{=} \bigcirc^i P$, with P a propositional atom.
- $TGI((B_1, B_2), i) \stackrel{\text{def}}{=} \{(B'_1, B'_2) \mid B'_1 \in TGI(B_1, i) \text{ and } B'_2 \in TGI(B_2, i)\}$.
- $TGI(\bigcirc B, i) \stackrel{\text{def}}{=} TGI(B, i+1)$.
- $TGI(\Diamond B, i) \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} TGI(B, i+k)$. ⊠

Then, we extend the definition of TGI for clauses in the following way.

Definition 16 (*TGI of a clause*)
We say that a temporal clause C^* is a *temporal ground instance* (TGI) of a temporal clause C iff:

(i) If $C = \bigcirc^i A \leftarrow B$ is an initial clause and $C^* = \bigcirc^i A \leftarrow B^*$ with $i \in \mathbb{N}$ and $B^* \in TGI(B, 0)$.
(ii) If $C = \Box \bigcirc^i A \leftarrow B$ is an initial clause and $C^* = \bigcirc^{i+k} A \leftarrow B^*$ with $k \in \mathbb{N}$ and $B^* \in TGI(B, 0)$.
(iii) If $C = \Box \left(\bigcirc^i A \leftarrow B\right)$ is a permanent clause and $C^* = \bigcirc^{i+k} A \leftarrow \bigcirc^k B^*$ with $k \in \mathbb{N}$ and $B^* \in TGI(B, 0)$. ⊠

Given a TEMPLOG program Π we define Π^* as the union of all temporal ground instances of clauses in Π. It is easy to see that Π^* is a positive program for signature At^∞ that can be understood as a classical propositional signature with atoms '$\bigcirc^i p$'. Therefore, Π^* has a least model that defines the TEMPLOG semantics.

This definition in terms of an infinite expansion of TGI's is a clear example where the infinitary HT translation can be applied: it is not difficult to see that the expansion results from applying distributive properties among infinite conjunctions and disjunctions. Indeed, we begin proving that the TGI's of a clause body have a clear translation into infinitary HT:

Proposition 1
Let $M^\infty = \langle H^\infty, T^\infty \rangle$ be an HT$^\infty$ interpretation and B a TEMPLOG body. Then:

$$M^\infty \models \{\alpha \mid \alpha \in TGI(B, i)\}^\vee \quad \text{iff} \quad M^\infty \models \langle B \rangle_i$$

where we assume that the comma ',' operator in each body in $TGI(B, i,)$ is interpreted as '$\wedge$' in HT$^\infty$. ⊠

We obtain next a similar infinitary encoding for program Π^*.

Theorem 4
Let Π be a TEMPLOG program for a signature At and $M^\infty = \langle H^\infty, T^\infty \rangle$ be an infinitary HT interpretation for that signature. Then:

$$M^\infty \models \Pi^* \quad \text{iff} \quad M^\infty \models \langle \Pi \rangle_0 \qquad \boxtimes$$

As a result, if we consider now the THT interpretation $\mathbf{M}$ corresponding to M^∞, we can use Theorem 3 to conclude that $M^\infty \models \Pi^*$ iff $\mathbf{M}, 0 \models \Pi$ in THT. Then, it is not difficult to prove that:

Theorem 5
Let Π be a TEMPLOG program and L the least model of Π^*. Then, the unique TS-model of Π is $\mathbf{T}$ defined as $T_i = \{p \mid \text{for any } \bigcirc^i p \in L\}$. $\boxtimes$

6 Conclusions

In this paper we have provided a pair of sound translations of Temporal Equilibrium Logic (TEL) into (a fragment of) Quantified Equilibrium Logic (QEL) and into Infinitary Equilibrium Logic. The correctness of these translations provide a solid justification for the semantics of TEL for arbitrary temporal theories. In the case of the translation into QEL, we have simply proved that Kamp's translation from LTL into monadic First Order Logic with linear order, FOL($\leq$), is also correct for translating TEL into QEL. This is also a good property confirming the adequacy of TEL semantics. An interesting open topic is whether the other direction of Kamp's theorem, i.e., that FOL($\leq$) can also be translated into LTL, also holds for translating back the corresponding QEL fragment into TEL. The study of this property is left for future work.

Regarding the infinitary translation, it provides a second encoding that can be more comfortable than QEL for some purposes, since the obtained expressions can be manipulated until a "propositional" program (formed with atoms preceded by next operators) is obtained. In fact, we have used this technique to prove that a temporal extension of Prolog, TEMPLOG, is subsumed by TEL, adding again one more justification for the suitability of TEL semantics.

Apart from the already mentioned completion of Kamp's theorem for TEL, future work will be focused on further exploring the infinitary encoding as a tool for proving other expressiveness properties of THT and TEL that remain open like, for instance, checking whether $\mathcal{U}$ can be expressed in terms of $\mathcal{R}$ and the other connectives, or checking whether the set of TS-models of a theory is representable with an LTL formula.

References

AGUADO, F., CABALAR, P., DIÉGUEZ, M., PÉREZ, G., AND VIDAL, C. 2013. Temporal equilibrium logic: a survey. *Journal of Applied Non-Classical Logics*. (to appear).

BALBIANI, P. AND DIÉGUEZ, M. 2015. An axiomatisation of the logic of temporal here-and-there. (unpublished draft).

BARAL, C. 2003. *Knowledge Representation, Reasoning and Declarative Problem Solving*. Elsevier.

BAUDINET, M. 1992. A simple proof of the completeness of temporal logic programming. In *Intensional Logics for Programming*, L. F. del Cerro and M. Penttonen, Eds. Clarendon Press, 51–83.

BIEBER, P., FARIÑAS DEL CERRO, L., AND HERZIG, A. 1988. MOLOG: a modal PROLOG. In *9th International Conference on Automated Deduction, Argonne, Illinois, USA, May 23-26, 1988, Proceedings.* 762–763.

BOZZELLI, L. AND PEARCE, D. 2015. On the complexity of temporal equilibrium logic. In *Proceedings of the 30th Annual ACM/IEEE Symposium of Logic in Computer Science (LICS'15)* (July 6-10). Kyoto, Japan. (to appear).

BREWKA, G., EITER, T., AND TRUSZCZYNSKI, M. 2011. Answer set programming at a glance. *Communications of the ACM 54,* 12, 92–103.

BÜCHI, J. R. 1962. On a Decision Method in Restricted Second Order Arithmetic. In *International Congress on Logic, Methodology, and Philosophy of Science*. 1–11.

CABALAR, P. AND DEMRI, S. 2011. Automata-based computation of temporal equilibrium models. In *21st International Symposium on Logic-Based Program Synthesis and Transformation (LOPSTR'11)*.

CABALAR, P. AND DIÉGUEZ, M. 2011. STELP - a tool for temporal answer set programming. In *LPNMR'11*. Lecture Notes in Computer Science, vol. 6645. 370–375.

CABALAR, P. AND DIÉGUEZ, M. 2014. Strong equivalence of non-monotonic temporal theories. In *Proceedings of the 14th International Conference on Principles of Knowledge Representation and Reasoning (KR'14)* (July 20-24). Vienna, Austria.

CABALAR, P. AND VEGA, G. P. 2007. Temporal equilibrium logic: a first approach. In *Proc. of the 11th International Conference on Computer Aided Systems Theory, (EUROCAST'07). LNCS (4739)*. 241–248.

FARIÑAS DEL CERRO, L. 1986. MOLOG: A system that extends PROLOG with modal logic. *New Generation Computing 4,* 1, 35–50.

FERRARIS, P. 2005a. Answer sets for propositional theories. In *Proc. of the 8th International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR'05). LNCS (vol. 3662)*. 119–131.

FERRARIS, P. 2005b. Answer sets for propositional theories. In *Proc. of the 8^{th} International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR'05)*. Diamante, Italy, 119–131.

FUJITA, M., KONO, S., TANAKA, T., AND MOTO-OKA, T. 1986. Tokio: Logic programming language based on temporal logic and its compilation into prolog. In *Proc. of the 3rd Intl. Conf. on Logic Programming (LNAI 225)*. Springer-Verlag.

GABBAY, D. 1987. Modal and temporal logic programming. In *Temporal Logics and their Applications*, A. Galton, Ed. Academic Press, Chapter 6, 197–237.

GELFOND, M. AND LIFSCHITZ, V. 1993. Representing action and change by logic programs. *Journal of Logic Programming 17,* 2/3&4, 301–321.

GELFOND, M. AND LIFSCHITZ, V. 1998. Action languages. *Linköping Electronic Articles in Computer and Information Science 3,* 16 (October).

HARRISON, A., LIFSCHITZ, V., PEARCE, D., AND VALVERDE, A. 2014. Infinitary equilibrium logic. In *Working Notes of Workshop on Answer Set Programming and Other Computing Paradigms (ASPOCP'14)*.

Kamp, J. A. 1968. Tense logic and the theory of linear order. Ph.D. thesis, University of California at Los Angeles.

Karp, C. R. 1964. *Languages with expressions of infinite length.* Studies in logic and the foundations of mathematics. North-Holland Publication Company.

Manna, Z. and Pnueli, A. 1991. *The Temporal Logic of Reactive and Concurrent Systems: Specification*. Springer-Verlag.

Marek, V. and Truszczyński, M. 1999. *Stable models and an alternative logic programming paradigm*. Springer-Verlag, 169–181.

Moszkowski, B. 1986. *Executing Temporal Logic Programs.* Cambridge University Press.

Niemelä, I. 1999. Logic programs with stable model semantics as a constraint programming paradigm. *Annals of Mathematics and Artificial Intelligence 25*, 241–273.

Orgun, M. A. and Wadge, W. W. 1992. Theory and practice of temporal logic programming. In *Intensional Logics for Programming*, L. F. del Cerro and M. Penttonen, Eds. Clarendon Press, Oxford, 21–50.

Pearce, D. 1996. A new logical characterisation of stable models and answer sets. In *Non monotonic extensions of logic programming. Proc. NMELP'96. (LNAI 1216)*. Springer-Verlag.

Pearce, D. 2006. Equilibrium logic. *Annals of Mathematics and Artificial Intelligence 47,* 1-2, 3–41.

Pearce, D. and Valverde, A. 2008. Quantified equilibrium logic and foundations for answer set programs. In *Proc. of the 24th Intl. Conf. on Logic Programming, ICLP 2008, (Udine, Italy, December 9-13)*. Lecture Notes in Computer Science, vol. 5366. Springer, 546–560.

Scott, D. and Tarski, A. 1958. The Sentential Calculus With Infinitely Long Expressions. *Colloquium Mathematicae 6,* 1, 165–170.

Truszczyński, M. 2012. Connecting First-Order ASP and the Logic FO(ID) through Reducts. In *Correct Reasoning - Essays on Logic-Based AI in Honour of Vladimir Lifschitz*. Lecture Notes in Computer Science, vol. 7265. Springer, 543–559.

van Emden, M. H. and Kowalski, R. A. 1976. The semantics of predicate logic as a programming language. *Journal of the ACM 23*, 733–742.

Appendix. Proofs

Proof of Theorem 1. We proceed by structural induction.

- If $\varphi = \bot$ then $[\varphi]_i = \bot$ and the result is straightforward.
- If $\varphi = p$ is an atom, then $[p]_i = p(i)$ and we get the chain of equivalent conditions: $\mathbf{M}, i \models p \Leftrightarrow p \in H_i \Leftrightarrow p(i) \in \mathcal{H} \Leftrightarrow \mathcal{M} \models p(i)$.
- If $\varphi = \alpha \wedge \beta$ we get:

$$\begin{array}{ll} & \mathbf{M}, i \models \alpha \wedge \beta \\ \Leftrightarrow & \mathbf{M}, i \models \alpha \text{ and } \mathbf{M}, i \models \alpha \\ \Leftrightarrow & \mathcal{M} \models [\alpha]_i \text{ and } \mathcal{M} \models [\beta]_i \qquad \text{by induction on } \alpha, \beta \\ \Leftrightarrow & \mathcal{M} \models [\alpha]_i \wedge [\beta]_i \\ \Leftrightarrow & \mathcal{M} \models [\alpha \wedge \beta]_i \end{array}$$

- The proof for $\varphi = \alpha \vee \beta$ is analogous to the one for $\alpha \wedge \beta$.
- If $\varphi = \alpha \to \beta$ we get:

$$\begin{array}{ll} & \mathbf{M}, i \models \alpha \to \beta \\ \Leftrightarrow & \text{for any } w \in \{\mathbf{H}, \mathbf{T}\}, \langle w, \mathbf{T} \rangle, i \not\models \alpha \text{ or } \langle w, \mathbf{T} \rangle, i \models \beta \end{array}$$

  Now, since the THT-interpretation $\langle \mathbf{T}, \mathbf{T} \rangle$ also corresponds to the MHT($\leq$) interpretation $\langle \mathcal{T}, \mathcal{T} \rangle$ we can apply induction on subformulas, so that we continue with the equivalent conditions:

$$\begin{array}{ll} \Leftrightarrow & \text{for any } w \in \{\mathcal{H}, \mathcal{T}\}, \langle w, \mathcal{T} \rangle \not\models [\alpha]_i \text{ or } \langle w, \mathcal{T} \rangle \models [\beta]_i \\ \Leftrightarrow & \langle \mathcal{H}, \mathcal{T} \rangle \models [\alpha \to \beta]_i. \end{array}$$

- If $\varphi = \bigcirc \alpha$ we get the equivalent conditions:

$$\begin{array}{ll} & \mathbf{M}, i \models \bigcirc \alpha \\ \Leftrightarrow & \mathbf{M}, i+1 \models \alpha \\ \Leftrightarrow & \mathcal{M} \models [\alpha]_{i+1} \qquad \text{by induction} \\ \Leftrightarrow & \mathcal{M} \models [\bigcirc \alpha]_i \end{array}$$

- If $\varphi = \alpha \, \mathcal{U} \, \beta$ we get the equivalent conditions:

$$\begin{array}{ll} & \mathbf{M}, i \models \alpha \, \mathcal{U} \, \beta \\ \Leftrightarrow & \begin{array}{l} \text{There is some } k \geq i \text{ s.t. that } \mathbf{M}, k \models \beta \text{ and} \\ \text{for } j \in \{i, \dots, k-1\}, \mathbf{M}, j \models \alpha \end{array} \\ \Leftrightarrow & \begin{array}{l} \text{There is some } k \geq i \text{ s.t. that } \mathcal{M} \models [\beta]_k \text{ and} \\ \text{for } j \in \{i, \dots, k-1\}, \mathcal{M} \models [\alpha]_j \end{array} \\ \Leftrightarrow & \mathcal{M} \models \exists k (i \leq k \wedge [\beta]_k \wedge \forall j (i \leq j < k \to [\alpha]_j)) \\ \Leftrightarrow & \mathcal{M} \models [\alpha \, \mathcal{U} \, \beta]_i \end{array}$$

- The proof for $\varphi = \alpha \, \mathcal{R} \, \beta$ is analogous to the one for $\alpha \, \mathcal{U} \, \beta$. ⊠

Proof of Theorem 2. We begin noting that the ordering relation between temporal interpretations $\mathbf{H} \leq \mathbf{T}$ is also in one-to-one correspondence with the ordering relation $\mathcal{H} \subseteq \mathcal{T}$ as sets of monadic predicate atoms. Then, the result follows from the definitions of TS-model and stable model, together with Theorem 1. ⊠

Proof of Theorem 3. To prove (i), we proceed by structural induction (by (Ind.) we denote the Induction Hypothesis):

$$
\begin{aligned}
\mathbf{M}, i \models p, \text{ with } p \in At &\Leftrightarrow p \in H_i \\
&\Leftrightarrow \bigcirc^i p \in H^\infty && \text{(Def. 14)} \\
&\Leftrightarrow M^\infty \models \langle p \rangle_i
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}, i \models \alpha \wedge \beta &\Leftrightarrow \mathbf{M}, i \models \alpha \text{ and } \mathbf{M}, i \models \beta \\
&\Leftrightarrow M^\infty \models \langle \alpha \rangle_i \text{ and } M^\infty \models \langle \beta \rangle_i && \text{(Ind.)} \\
&\Leftrightarrow M^\infty \models \{\langle \alpha \rangle_i, \langle \beta \rangle_i\}^\wedge && \text{(Def. 11)} \\
&\Leftrightarrow M^\infty \models \langle \alpha \wedge \beta \rangle_i && \text{(Def. 13)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}, i \models \alpha \vee \beta &\Leftrightarrow \mathbf{M}, i \models \alpha \text{ or } \mathbf{M}, i \models \beta \\
&\Leftrightarrow M^\infty \models \langle \alpha \rangle_i \text{ or } M^\infty \models \langle \beta \rangle_i && \text{(Ind.)} \\
&\Leftrightarrow M^\infty \models \{\langle \alpha \rangle_i, \langle \beta \rangle_i\}^\vee && \text{(Def. 11)} \\
&\Leftrightarrow M^\infty \models \langle \alpha \vee \beta \rangle_i && \text{(Def. 13)}
\end{aligned}
$$

$$\mathbf{M}, i \models \alpha \rightarrow \beta \Leftrightarrow \text{ for any } w \in \{\mathbf{H}, \mathbf{T}\}, \langle w, \mathbf{T} \rangle, i \not\models \alpha \text{ or } \langle w, \mathbf{T} \rangle, i \models \beta$$

Now, since the THT-interpretation $\langle \mathbf{T}, \mathbf{T} \rangle$ also corresponds to the HT^∞ interpretation $\langle T, T \rangle$, we can apply induction on subformulas, so that we continue with the equivalent conditions:

$$
\begin{aligned}
&\Leftrightarrow \text{ for any } w \in \{H^\infty, T^\infty\}, \langle w, T^\infty \rangle \not\models \langle \alpha \rangle_i \text{ or } \langle w, T^\infty \rangle \models \langle \beta \rangle_i && \text{(Ind.)} \\
&\Leftrightarrow M^\infty \models \langle \alpha \rightarrow \beta \rangle_i .
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}, i \models \alpha \mathcal{U} \beta &\Leftrightarrow \exists j,\ i \leq j \text{ s.t. } \mathbf{M}, j \models \beta \text{ and } \forall k,\ \text{if } i \leq k < j \text{ then } \mathbf{M}, k \models \alpha \\
&\Leftrightarrow \exists j,\ i \leq j \text{ s.t. } M^\infty, \models \langle \beta \rangle_j \text{ and } \forall k,\ \text{if } i \leq k < j \text{ then } M^\infty \models \langle \alpha \rangle_k && \text{(Ind.)} \\
&\Leftrightarrow M^\infty \models \{\{\langle \alpha \rangle_k, \langle \beta \rangle_j \mid i \leq k < j\}^\wedge \mid i \leq j\}^\vee && \text{(Def. 11)} \\
&\Leftrightarrow M^\infty \models \langle \alpha \mathcal{U} \beta \rangle_i && \text{(Def. 13)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}, i \models \alpha \mathcal{R} \beta &\Leftrightarrow \forall j,\ j \leq i \text{ if } \mathbf{M}, j \not\models \beta \text{ then } \exists k,\ i \leq k < j \text{ such that } M, j \models \alpha \\
&\Leftrightarrow \forall j,\ j \leq i \text{ if } M^\infty \not\models \langle \beta \rangle_j \text{ then } \exists k,\ i \leq k < j \text{ such that } M^\infty \models \langle \alpha \rangle_k && \text{(Ind.)} \\
&\Leftrightarrow M^\infty \models \{\{\langle \beta \rangle_j, \langle \alpha \rangle_k \mid i \leq k < j\}^\vee \mid i \leq j\}^\wedge && \text{(Def. 11)} \\
&\Leftrightarrow M^\infty \models \langle \alpha \mathcal{R} \beta \rangle_i && \text{(Def. 13)}
\end{aligned}
$$

The proof for (ii) is straightforward since there exists a one-to-one correspondence between ordering relations among temporal and infinitary interpretations. ⊠

Proof of Proposition 1. We proceed by structural induction. The proof for the base case comes from the definition of the translation.

$$
\begin{aligned}
M^\infty \models TGI(\epsilon, i)^\vee &\Leftrightarrow M^\infty \models \emptyset^\vee \\
&\Leftrightarrow M^\infty \models \langle \bot \rangle_i && \text{(Def. 13)}
\end{aligned}
$$

$$
\begin{aligned}
M^\infty \models TGI(P, i)^\vee &\Leftrightarrow M^\infty \models \bigcirc^i P \\
&\Leftrightarrow M^\infty \models \langle P \rangle_i && \text{(Def. 13)}
\end{aligned}
$$

Now we continue with the inductive step. For the case of conjunction, if we apply Definition 15, we get

$$M^\infty \models TGI(B_1, B_2, i)^\vee \Leftrightarrow M^\infty \models \{B_1^{'}, B_2^{'} \mid B_1^{'} \in TGI(B_1, i)^\vee \text{ and } B_2^{'} \in TGI(B_2, i)^\vee\}^\vee$$

which, because of the distributivity property of infinitary HT, it corresponds to

$$M^\infty \models TGI(B_1, B_2, i)^\vee \Leftrightarrow M^\infty \models TGI(B_1, i)^\vee \text{ and } M^\infty \models TGI(B_2, i)^\vee$$

and the chain of equivalences would continue as follows:

$$\begin{aligned} M^\infty \models TGI(B_1, i)^\vee \text{ and } M^\infty \models TGI(B_2, i)^\vee &\Leftrightarrow M^\infty \models \langle B_1 \rangle_i \text{ and } M^\infty \models \langle B_2 \rangle_i && \text{(Ind.)} \\ &\Leftrightarrow M^\infty \models \langle B_1, B_2 \rangle_i && \text{(Def. 13)} \end{aligned}$$

$$\begin{aligned} M^\infty \models TGI(\bigcirc B, i)^\vee &\Leftrightarrow M^\infty \models TGI(B, i+1)^\vee && \text{(Def. 15)} \\ &\Leftrightarrow M^\infty \models \langle B \rangle_{i+1} && \text{(Ind.)} \\ &\Leftrightarrow M^\infty \models \langle \bigcirc B \rangle_i && \text{(Def. 13)} \end{aligned}$$

$$\begin{aligned} M^\infty \models TGI(\Diamond B, i)^\vee &\Leftrightarrow M^\infty \models \{B^{'} \mid B^{'} \in TGI(B, i+k)\}^\vee && \text{(Def. 15)} \\ &\Leftrightarrow \exists\, k \geq 0 \text{ and } B^{'} \in TGI(B, i+k)^\vee \text{ s.t. } M^\infty \models B^{'} && \text{(Def. 11)} \\ &\Leftrightarrow M^\infty \models TGI(B, i+k)^\vee, \text{ with } k \geq 0 && \text{(Def. 11)} \\ &\Leftrightarrow M^\infty \models \langle B \rangle_{i+k} && \text{(Ind.)} \\ &\Leftrightarrow M^\infty \models \langle \Diamond B \rangle_i && \text{(Def. 13)} \end{aligned}$$

Proposition 2
The following equivalences are valid in Infinitary Here and There

$$\mathcal{H}^\vee \to \beta \quad \Leftrightarrow \quad \{\alpha \to \beta \mid \alpha \in \mathcal{H}\}^\wedge \tag{8}$$

$$\alpha \to \mathcal{H}^\wedge \quad \Leftrightarrow \quad \{\alpha \to \beta \mid \beta \in \mathcal{H}\}^\wedge \tag{9}$$

Proof

$$\begin{aligned} & \langle H, T \rangle \models \mathcal{H}^\vee \to \beta \\ \Leftrightarrow\ & \langle H, T \rangle \not\models \mathcal{H}^\vee \text{ or } \langle H, T \rangle \models \beta \text{ and } \langle T, T \rangle \not\models \mathcal{H}^\vee \text{ or } \langle T, T \rangle \models \beta && \text{(Def. 11)} \\ \Leftrightarrow\ & \forall \alpha \in \mathcal{H},\ \langle H, T \rangle \not\models \alpha \text{ or } \langle H, T \rangle \models \beta \text{ and } \forall \alpha \in \mathcal{H},\ \langle T, T \rangle \not\models \alpha \text{ or } \langle T, T \rangle \models \beta && \text{(Def. 11)} \\ \Leftrightarrow\ & \forall \alpha \in \mathcal{H},\ \langle H, T \rangle \models \alpha \to \beta \text{ and } \forall \alpha \in \mathcal{H},\ \langle T, T \rangle \models \alpha \to \beta \\ \Leftrightarrow\ & \langle H, T \rangle \models \{\alpha \to \beta \mid \alpha \in \mathcal{H}\}^\wedge && \text{(Def. 11)} \end{aligned}$$

$$\begin{aligned} & \langle H, T \rangle \models \alpha \to \mathcal{H}^\wedge \\ \Leftrightarrow\ & \langle H, T \rangle \not\models \alpha \text{ or } \langle H, T \rangle \models \mathcal{H}^\wedge \text{ and } \langle T, T \rangle \not\models \alpha \text{ or } \langle T, T \rangle \models \mathcal{H}^\wedge && \text{(Def. 11)} \\ \Leftrightarrow\ & \langle H, T \rangle \not\models \alpha \text{ or } \forall \beta \in \mathcal{H},\ \langle H, T \rangle \models \beta \text{ and } \langle T, T \rangle \not\models \alpha \text{ or } \forall \beta \in \mathcal{H},\ \langle T, T \rangle \models \beta && \text{(Def. 11)} \\ \Leftrightarrow\ & \forall \beta \in \mathcal{H},\ \langle H, T \rangle \not\models \alpha \text{ or } \langle H, T \rangle \models \beta \text{ and } \forall \beta \in \mathcal{H},\ \langle T, T \rangle \not\models \alpha \text{ or } \langle T, T \rangle \models \beta \\ \Leftrightarrow\ & \forall \beta \in \mathcal{H},\ \langle H, T \rangle \models \alpha \to \beta \text{ and } \forall \beta \in \mathcal{H},\ \langle T, T \rangle \models \alpha \to \beta \\ \Leftrightarrow\ & \langle H, T \rangle \models \{\alpha \to \beta \mid \beta \in \mathcal{H}\}^\wedge && \text{(Def. 11)} \end{aligned}$$

□

Proof of Theorem 4. We proceed by checking cases (i)-(iii) of Definition 16.

$$
\begin{aligned}
M^\infty \models (i) &\Leftrightarrow \forall B^* \in TGI(B,0),\ M^\infty \models \bigcirc^i A \leftarrow B^* \\
&\Leftrightarrow M^\infty \models \{\bigcirc^i A \leftarrow B^* \mid B^* \in TGI(B,0)\}^\wedge && \text{(Def. 11)} \\
&\Leftrightarrow M^\infty \models \bigcirc^i A \leftarrow \{B^* \mid B^* \in TGI(B,0)\}^\vee && \text{(Prop. 2)} \\
&\Leftrightarrow M^\infty \models \langle \bigcirc^i A\rangle_0 \leftarrow \langle B\rangle_0 && \text{(Prop. 1)} \\
&\Leftrightarrow M^\infty \models \langle \bigcirc^i A \leftarrow B\rangle_0 && \text{(Def. 13)}
\end{aligned}
$$

$$
\begin{aligned}
M^\infty \models (ii) &\Leftrightarrow \forall k \geq 0\ \forall B^* \in TGI(B,0),\ M^\infty \models \bigcirc^{i+k} A \leftarrow B^* \\
&\Leftrightarrow \begin{cases} M^\infty \models \{\{\bigcirc^{i+k} A \leftarrow B^* \mid B^* \in TGI(B,0)\}^\wedge \\ \qquad\qquad\qquad \mid k \geq 0\}^\wedge \end{cases} && \text{(Def. 11)} \\
&\Leftrightarrow \begin{cases} M^\infty \models \{\bigcirc^{i+k} A \leftarrow \{B^* \mid B^* \in TGI(B,0)\}^\vee \\ \qquad\qquad\qquad \mid k \geq 0\}^\wedge \end{cases} && \text{(Prop. 2)} \\
&\Leftrightarrow M^\infty \models \{\bigcirc^{i+k} A \mid k \leq 0\}^\wedge \leftarrow \{B^* \mid B^* \in TGI(B,0)\}^\vee && \text{(Prop. 2)} \\
&\Leftrightarrow M^\infty \models \{\bigcirc^{i+k} A \mid k \leq 0\}^\wedge \leftarrow \langle B\rangle_0 && \text{(Prop. 1)} \\
&\Leftrightarrow M^\infty \models \langle \Box \bigcirc^i A\rangle_0 \leftarrow \langle B\rangle_0 && \text{(Def. 13)} \\
&\Leftrightarrow M^\infty \models \langle \Box \bigcirc^i A \leftarrow B\rangle_0 && \text{(Def. 13)}
\end{aligned}
$$

$$
\begin{aligned}
M^\infty \models (iii) &\Leftrightarrow \forall k \geq 0\ \forall B^* \in TGI(B,0),\ M^\infty \models \bigcirc^{i+k} A \leftarrow \bigcirc^k B^* \\
&\Leftrightarrow \begin{cases} M^\infty \models \{\{\bigcirc^{i+k} A \leftarrow \bigcirc^k B^* \mid B^* \in TGI(B,0)\}^\wedge \\ \qquad\qquad\qquad \mid k \geq 0\}^\wedge \end{cases} && \text{(Def. 11)} \\
&\Leftrightarrow \begin{cases} M^\infty \models \{\bigcirc^{i+k} A \leftarrow \{\bigcirc^k B^* \mid B^* \in TGI(B,0)\}^\vee \\ \qquad\qquad\qquad \mid k \geq 0\}^\wedge \end{cases} && \text{(Prop. 2)} \\
&\Leftrightarrow M^\infty \models \{\bigcirc^{i+k} A \leftarrow \{\widehat{B} \mid \widehat{B} \in TGI(B,k)\}^\vee \mid k \geq 0\}^\wedge && \text{(Def. 16)} \\
&\Leftrightarrow M^\infty \models \{\bigcirc^{i+k} A \leftarrow \langle B\rangle_k \mid k \geq 0\}^\wedge && \text{(Prop. 1)} \\
&\Leftrightarrow M^\infty \models \{\langle \bigcirc^{i+k} A\rangle_0 \leftarrow \langle B\rangle_k \mid k \geq 0\}^\wedge && \text{(Def. 13)}^5 \\
&\Leftrightarrow M^\infty \models \{\langle \bigcirc^{i} A\rangle_k \leftarrow \langle B\rangle_k \mid k \geq 0\}^\wedge && \text{(Def. 13)} \\
&\Leftrightarrow M^\infty \models \{\langle \bigcirc^{i} A \leftarrow B\rangle_k \mid k \geq 0\}^\wedge && \text{(Def. 13)} \\
&\Leftrightarrow M^\infty \models \langle \Box \left(\bigcirc^{i} A \leftarrow B\right)\rangle_0 && \text{(Def. 13)}
\end{aligned}
$$

Proof of Theorem 5. Since Π^* is a definite positive program, its least model L is its unique stable model. Next, from (Truszczyński 2012) we derive that L is the unique infinitary stable model and, moreover, $\langle L, L\rangle$ is the unique infinitary equilibrium model (this result was also proved in (Harrison et al. 2014)). Finally, the fact that $\langle L, L\rangle$ is the unique stable model of $\langle \Pi\rangle_0$ follows from Theorem 4 and, because of Lemma 14, $\langle \mathbf{T}, \mathbf{T}\rangle$ is the unique stable model of Π. ☒

[5] Note that, since A is an atom, then $\langle \bigcirc^{i+k} A\rangle_0 \equiv \bigcirc^{i+k} A$