A Fixpoint Characterisation of Temporal Equilibrium Logic

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Abstract. Connections of intuitionistic and intermediate logics with logic programming have been extensively studied in the literature. Among the different results in the literature we find Equilibrium Logic (Pearce, 1996) and Safe beliefs (Osorio et al, 2005). Pearce's approach admits a characterisation in terms of a fixpoint (consequence) operator on the here-and-there intermediate logic (Heyting, 1930), which is similar to the notion of *theory completion* in default and autoepistemic logics. Osorio's safe beliefs are also given in terms of a fixpoint operator under intuitionistic logic semantics. In this latter case, intuitionistic logic can be replaced by any intermediate logic without altering the result.
In this paper we consider temporal equilibrium logic, an extension of equilibrium logic with operators from Linear-time temporal logic. In this context we extend Pearce's and Osorio's approach to temporal case and we discuss the relation of intuitionistic temporal logic and temporal logic programming.

1 Introduction

Introduced in the late 80s, *Temporal Logic Programming* [1] refers to the extension of logic programming that incorporates modal temporal operators, mainly from *Linear-time Temporal Logic* [22] (LTL). Although a great deal of research was done in this field during the 80s and 90s, the topic lost interest over time. With the advent of *Answer Set Programming* [15] (ASP) and its use in the description and solving of temporal scenarios, those old ideas regained the interest of the logic programming community.

First approaches of time representation in ASP, which followed [10], represented time by means of a variable whose values were taken from a finite subset of $\mathbb{N}$. A major disadvantage of this representation is that it provides neither special language constructs nor specific inference methods for temporal reasoning, as available in LTL. As a result, it is infeasible to represent properties of reactive systems (with infinite runs or traces) such as safety ("Is some particular state reachable?"), liveness ("Does some condition happen infinitely often?"), or to conclude that a given planning problem has no solution at all.

Several extensions of ASP with temporal operators were investigated in the literature. All of them start by considering a temporal (or dynamic) modal logic [22, 13] and then they induce the non-monotonicity by combining it with one of the different semantics for ASP that are available in the literature [16].

For instance, Giordano et al. [12] have extended the traditional reduct-based semantics for logic programs [11] to a fragment of logic programming with operators from dynamic logic [13], while Cabalar et al [2] have extended LTL with Equilibrium Logic [20] (EL), the most popular logical characterisation of stable models and answer sets. This latter characterisation has been later extended to the case of finite traces [3].

Equilibrium Logic is defined in terms of the Here-and-There logic [14] (HT) together with a minimal model selection criterion that selects the minimal models (or answer sets). Moreover, in his seminal paper [20], Pearce also characterized Equilibrium Logic in a more syntactical style, as a kind of fixpoint logic. The idea consists of considering theory extensions (or *completions*) of a theory instead of finding minimal models, as happens in default and autoepistemic logics [17].

A slightly different fixpoint characterisation was given by M. Osorio et al. [19] called safe beliefs. This characterisation, uses Intuitionistic Logic [18] (IL) as underlying monotonic basis and, moreover, they have proven that IL can be replaced by any logic X satisfying IL $\subseteq$ X $\subseteq$ HT without changing the set of safe beliefs. As a result, safe beliefs provide us with a framework for studying properties of logic programs from a more general point of view.

In this paper, we extend Pearce's and Osorio's characterizations to the temporal case. In the case of Pearce's approach, we use directly THT as an underlying logic while, in the case of Osorio's, we provide a more general definition in terms of the *intuitionistic temporal logic interpreted on persistent* frames [5] ($\mathrm{ITL}^{\mathrm{p}}$) and then show that when replacing $\mathrm{ITL}^{\mathrm{p}}$ by THT the safe beliefs coincide with the temporal equilibrium models.

This paper is organised as follows: Section 2 introduces the family of intuitionistic temporal logics. In this section we discuss the different intuitionistic temporal logics that can be defined by combining different confluence properties and we also define the concept of intermediate temporal logics. In section 3 we introduce temporal equilibrium logic, which is the temporal logic programming semantics we will consider along this paper. Section 4 we present both Pearce's and Osorio's fixpoint characterisations of equilibrium and stable models. These characterisations are extended to the temporal case in Section 5. Finally, in Section 6 we present the conclusions and several lines of future work.

2 Intuitionistic Temporal Logics and Intermediate Logics

Given a (countable, possibly infinite) set $\mathbb{P}$ of atoms (called *alphabet*), the temporal language $\mathcal{L}$ consists of formulas generated by the following grammar:

$$\varphi ::= p \mid \bot \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \rightarrow \varphi_2 \mid \circ\varphi \mid \varphi_1 \mathbin{\mathbb{U}} \varphi_2 \mid \varphi_1 \mathbin{\mathbb{R}} \varphi_2$$

where $p \in \mathbb{P}$ is an atom. The intended meaning of the previous temporal operators is the following: $\circ\varphi$ means that φ is true at the next time point. $\varphi \mathbin{\mathbb{U}} \psi$ means that φ is true until ψ is true. For $\varphi \mathbin{\mathbb{R}} \psi$ the meaning is not as direct as for the previous operators. $\varphi \mathbin{\mathbb{R}} \psi$ means that ψ is true until both φ and ψ

become true simultaneously or ψ is true forever. We also define several common derived operators like the Boolean connectives $\top \stackrel{def}{=} \neg\bot$, $\neg\varphi \stackrel{def}{=} \varphi \to \bot$, $\varphi \leftrightarrow \psi \stackrel{def}{=} (\varphi \to \psi) \wedge (\psi \to \varphi)$, and the unary temporal operators $\Box\varphi \stackrel{def}{=} \bot \mathbin{\mathbb{R}} \varphi$ (*always afterwards*) and $\Diamond\varphi \stackrel{def}{=} \top \mathbin{\mathbb{U}} \varphi$. A *(temporal) theory* is a (possibly infinite) set of temporal formulas. Formulas of $\mathcal{L}$ are interpreted over dynamic posets. A *dynamic poset* is a tuple $D = (W, \preccurlyeq, S)$, where W is a non-empty set of (Kripke) worlds, $\preccurlyeq$ is a partial order, and S is a function from W to W satisfying the *forward confluence* condition that for all $w, v \in W$, if $w \preccurlyeq v$ then $S(w) \preccurlyeq S(v)$. Moreover, if S satisfies the *backward confluence* condition that for all $w, v, u \in W$ if $S(w) = v \preccurlyeq u$ then there exists $t \in W$ such that $w \preccurlyeq t$ and $S(t) = u$. Figure 1 shows a graphical version of the aforementioned confluence properties: Figure 1a corresponds to the forward confluence while Figure 1b corresponds to the backward confluence relation. On a dynamic poset the above diagrams can always be completed if S is forward or backward confluent (represented by means of dashed arrows), respectively. Posets with both properties are *persistent.*

$$\begin{tikzcd} v \arrow[r, "S"] & S(v) \\ w \arrow[u, "\curlyvee"'] \arrow[r, "S"'] & S(w) \arrow[u, dashed, "\curlyvee"'] \end{tikzcd}$$

(a) Forward confluence

$$\begin{tikzcd} t \arrow[r, dashed, "S"] & u \\ w \arrow[u, dashed, "\curlyvee"] \arrow[r, "S"'] & v \arrow[u, "\curlyvee"'] \end{tikzcd}$$

(b) Backward confluence

Fig. 1: Diagrams associated to the forward and backward confluence conditions

An *intuitionistic dynamic model,* or simply *model,* is a tuple $\mathfrak{M} = (W, \preccurlyeq, S, V)$ consisting of a dynamic poset equipped with a valuation function $V : W \to \wp(\mathbb{P})$ that is *monotone* in the sense that for all $w, v \in W$, if $w \preccurlyeq v$ then $V(w) \subseteq V(v)$. In the standard way, we define $S^0(w) = w$ and, for all $k \geq 0$, $S^{k+1}(w) = S\left(S^k(w)\right)$. Then we define the satisfaction relation $\models$ inductively by:

1. $\mathfrak{M}, w \models p$ iff $p \in V(w)$;
2. $\mathfrak{M}, w \not\models \bot$;
3. $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$;
4. $\mathfrak{M}, w \models \varphi \vee \psi$ iff $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$;
5. $\mathfrak{M}, w \models \circ\varphi$ iff $\mathfrak{M}, S(w) \models \varphi$;
6. $\mathfrak{M}, w \models \varphi \to \psi$ iff $\forall v \succcurlyeq w$, if $\mathfrak{M}, v \models \varphi$, then $\mathfrak{M}, v \models \psi$;
7. $\mathfrak{M}, w \models \Diamond\varphi$ iff there exists $k \geq 0$ such that $\mathfrak{M}, S^k(w) \models \varphi$;
8. $\mathfrak{M}, w \models \Box\varphi$ iff for all $k \geq 0$ we have that $\mathfrak{M}, S^k(w) \models \varphi$;
9. $\mathfrak{M}, w \models \varphi \mathbin{\mathbb{U}} \psi$ iff there exists $k \geq 0$ such that $\mathfrak{M}, S^k(w) \models \psi$ and $\forall i \in [0, k)$, $\mathfrak{M}, S^i(w) \models \varphi$;
10. $\mathfrak{M}, w \models \varphi \mathbin{\mathbb{R}} \psi$ iff for all $k \geq 0$, either $\mathfrak{M}, S^k(w) \models \psi$ or $\exists i \in [0, k)$ such that $\mathfrak{M}, S^i(w) \models \varphi$.

Figure 2 illustrates the '$\models$' relation. The reader may verify that $\mathfrak{M}, x \models \circ p$ but $\mathfrak{M}, x \not\models p$, while $\mathfrak{M}, y \models p$ but $\mathfrak{M}, y \not\models \circ p$. From this it follows that $\mathfrak{M}, w \not\models (\circ p \to p) \lor (p \to \circ p)$.

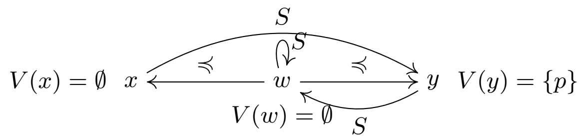

Fig. 2: Example of an ITL$^{\mathrm{e}}$ model $\mathfrak{M} = (W, \preccurlyeq, S, V)$, where reflexivity and transitivity for $\preccurlyeq$ are not represented.

A formula φ is *satisfiable in a logic X* if there is a model $\mathfrak{M}$ and a world w so that $\mathfrak{M}, w \models_{\mathrm{X}} \varphi$, and *valid in X* if, for every world w of every model $\mathfrak{M}$ we have that $\mathfrak{M}, w \models_{\mathrm{X}} \varphi$. Similarly, we say that a theory Γ is *consistent* within a logic X if there is a model $\mathfrak{M}$ and a world w such that $\mathfrak{M}, w \models_{\mathrm{X}} \varphi$, for all $\varphi \in \Gamma$, where $\models_{\mathrm{X}}$ stands for the satisfaction relation in the logic X. The logic X will be omitted from the satisfaction relation when clear from the context. When Γ is consistent in the logic X, we will say that Γ is X-consistent.

We will call *expanding domain intuitionisitic temporal logic* (ITL$^{\mathrm{e}}$) to the intuitionistic temporal logic interpreted over the class of intuitionistic dynamic models while we will denote by *persistent intuitionistic temporal logic* (ITL$^{\mathrm{p}}$) to the intuitionistic temporal logic interpreted over the class of the intuitionistic persistent models, i.e. those that satisfy the backward confluence property. First, we remark that dynamic posets impose the minimal conditions on S and $\preccurlyeq$ in order to preserve the monotonicity of truth of formulas, in the sense that if $\mathfrak{M}, w \models \varphi$ and $w \preccurlyeq v$ then $\mathfrak{M}, v \models \varphi$.

Proposition 1 ([5]). *Let $\mathfrak{D} = (W, \preccurlyeq, S)$, where $(W, \preccurlyeq)$ is a poset and $S\colon W \to W$ is any function. Then, the following are equivalent:*

1. *S is forward confluent;*
2. *for every valuation V on $\mathfrak{D}$ and every formula φ, truth of φ is monotone with respect to $\preccurlyeq$.*

However, satisfiability is an interesting and desired property that does not hold in ITL$^{\mathrm{e}}$: satisfiability in propositional intuitionistic logic is equivalent to satisfiability in classical propositional logic. This is because, if φ is classically satisfiable, it is trivially intuitionistically satisfiable in a one-world model; conversely, if φ is intuitionistically satisfiable, it is satisfiable in a finite model, hence in a maximal world of that finite model, and the generated submodel of a maximal world is a classical model. Thus it may be surprising that the same is not the case for intuitionistic temporal logic. To show this fact, let us consider the temporal formula $\varphi = \neg \circ p \land \neg \circ \neg p$ [5]. In classical logic, this formula is equivalent to $\circ \neg p \land \circ p$, which is not satisfiable. However, the reader can check that the ITL$^{\mathrm{e}}$ model $\mathfrak{M}$

shown in Figure 3 satisfies φ at the point w (in symbols, $\mathfrak{M}, w \models \varphi$). Morevoer, note that, in this case, S is forward, but not backward, confluent. Hence the decidability of the intuitionistic satisfiability problem is not a corollary of the classical case.

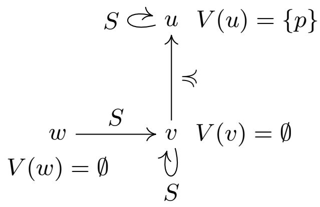

Fig. 3: A dynamic intuitionistic model, satisfying $\varphi = \neg\circ p \wedge \neg\circ\neg p$. Reflexivity and transitivity of $\preccurlyeq$ are omitted for the sake of clarity.

Proposition 2 ([5]). *The following formulas are* ITL$^{\mathrm{e}}$*-valid:*

1. $\circ\bot \leftrightarrow \bot$
2. $\circ(\varphi \wedge \psi) \leftrightarrow (\circ\varphi \wedge \circ\psi)$
3. $\circ(\varphi \vee \psi) \leftrightarrow (\circ\varphi \vee \circ\psi)$
4. $\circ(\varphi \to \psi) \to (\circ\varphi \to \circ\psi)$
5. $\circ\Box\varphi \leftrightarrow \Box\circ\varphi$
6. $\circ\Diamond\varphi \leftrightarrow \Diamond\circ\varphi$
7. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
8. $\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$;
9. $\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$;
10. $\Box\varphi \leftrightarrow \varphi \wedge \circ\Box\varphi$
11. $\varphi \vee \circ\Diamond\varphi \leftrightarrow \Diamond\varphi$
12. $\Box(\varphi \to \circ\varphi) \to (\varphi \to \Box\varphi)$
13. $\Box(\circ\varphi \to \varphi) \to (\Diamond\varphi \to \varphi)$

Proposition 3 ([5]). *The formula* $(\Diamond p \to \Box q) \to \Box(p \to q)$ *is not* ITL$^{\mathrm{e}}$*-valid but it is* ITL$^{\mathrm{p}}$*-valid. Moreover, when added as an axiom, it characterises the class of persistent dynamic posets.*

To the best of our knowledge the family of *intermediate temporal logics has not been defined in the literature.* In this paper, we are interested in defining such family as logics extending ITL$^{\mathrm{p}}$, as stated in the following definition.

Definition 1 (adapted from [9]). *An* intermediate temporal logic *in the language* $\mathcal{L}$ *is any set of formulas* S *satisfying the following conditions:*

1) ITL$^{\mathrm{p}}$ $\subseteq S$;
2) S *is closed under* modus ponens, *i.e.,* $\varphi, \varphi \to \psi \in S$ *implies* $\psi \in S$;
3) S *is closed under* uniform substitution, *i.e.,* $\varphi \in S$ *implies* $\varphi\mathbf{s} \in S$ *for any* $\varphi \in \mathcal{L}$ *and a substitution* $\mathbf{s}$.

Broadly speaking, intermediate temporal logics are obtained by adding axioms that force additional properties on the accessibility relations in ITL$^{\mathrm{p}}$ models. The reason why we have chose ITL$^{\mathrm{p}}$ as the “weakest” intuitionistic temporal logic is the following.

Proposition 4 ([5]). *Let X and Y be two logics satisfying* $\mathrm{ITL}^{\mathrm{p}} \subseteq X \subseteq Y$. *A theory Γ is X-consistent iff Γ is Y-consistent.*

Proposition 4 states that consistency is preserved along the different intermediate temporal logics. Such result will be used in Section 5. Also note that THT is the strongest intermediate temporal logic which is strictly included in classical logic [9].

An important logic that can be characterised in terms of persistent dynamic posets is the temporal here-and-there logic [4], as presented in the following definition.

Definition 2 ([4]). *A persistent dynamic poset $D = (W, \preccurlyeq, S)$ is said to be a HT persistent dynamic poset if it satisfies the following condition: for all $x,\ y,\ z \in W$, if $x \preccurlyeq y$ and $x \preccurlyeq z$ then either $x = y$ or $x = z$ or $y = z$;*

As in the case of $\mathrm{ITL}^{\mathrm{e}}$and $\mathrm{ITL}^{\mathrm{p}}$, a HT persistent dynamic poset equipped with a valuation V is called a HT *persistent dynamic model.* We will denote by $\mathrm{ITL}^{\mathrm{ht}}$ the intuitionistic temporal logic interpreted over the class of HT persistent dynamic models.

Proposition 5 ([5, 4]). *The logic* $\mathrm{ITL}^{\mathrm{ht}}$*is obtained by adding Hosoi's axiom $p \vee (p \rightarrow q) \vee \neg q$ to* $\mathrm{ITL}^{\mathrm{p}}$.

3 Temporal Equilibrium Logic

The definition of *(Linear-time) Temporal Equilibrium Logic* (TEL) is done in two steps. First, we define a monotonic logic called *(Linear-time) Temporal Here-and-There* (THT), a temporal extension of the intermediate logic of Here-and-There [14]. In a second step, we select some models from THT that are said to be *in equilibrium*, obtaining in this way a non-monotonic entailment relation.

In LTL, semantics relies on the concept of a *trace* over an alphabet $\mathbb{P}$, which is an infinite sequence $\mathbf{T} = (T_i)_{i \geq 0}$ of sets $T_i \subseteq \mathbb{P}$.

To represent a given trace, we use ω-regular expressions like, for instance, in the infinite trace $(\{a\} \cdot \emptyset)^\omega$ where all even positions make a true and all odd positions make it false.

At each state T_i in a trace, an atom a can only be true, viz. $a \in T_i$, or false, $a \notin T_i$. The logic THT weakens this truth assignment, following the same intuitions as the (non-temporal) logic of HT. In THT, an atom can have one of three truth-values in each state, namely, *false*, *assumed* (or true by default) or *proven* (or certainly true). Anything proved has to be assumed, but the opposite does not necessarily hold. Following this idea, a state i is represented as a pair of sets of atoms $\langle H_i, T_i \rangle$ with $H_i \subseteq T_i \subseteq \mathbb{P}$ where H_i (standing for "here") contains the proven atoms, whereas T_i (standing for "there") contains the assumed atoms. On the other hand, false atoms are just the ones not assumed, captured by $\mathbb{P} \setminus T_i$. Accordingly, a *Here-and-There trace* (for short *HT-trace*) of length λ over alphabet $\mathbb{P}$ is a sequence of pairs $(\langle H_i, T_i \rangle)_{i \in [0..\lambda)}$ with $H_i \subseteq T_i$ for $i \geq 0$.

For convenience, we usually represent the HT-trace as the pair $\langle \mathbf{H}, \mathbf{T} \rangle$ of traces $\mathbf{H} = (H_i)_{i \geq 0}$ and $\mathbf{T} = (T_i)_{i \geq 0}$. Note that the two traces $\mathbf{H}$, $\mathbf{T}$ must satisfy a kind of order relation, since $H_i \subseteq T_i$ for each time point i. Formally, we define the ordering $\mathbf{H} \leq \mathbf{T}$ between two traces as $H_i \subseteq T_i$ for each $i \geq 0$. Furthermore, we define $\mathbf{H} < \mathbf{T}$ as both $\mathbf{H} \leq \mathbf{T}$ and $\mathbf{H} \neq \mathbf{T}$. Thus, an HT-trace can also be defined as any pair $\langle \mathbf{H}, \mathbf{T} \rangle$ of traces such that $\mathbf{H} \leq \mathbf{T}$. The particular type of HT-traces satisfying $\mathbf{H} = \mathbf{T}$ are called *total*.

We proceed by generalizing the extension of HT with temporal operators, called THT [2], to HT-traces of fixed length in order to integrate finite as well as infinite traces. Given any HT-trace $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$, we define the THT satisfaction of formulas as follows.

Definition 3 (THT-satisfaction; [2]). *An HT-trace* $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ *over alphabet* $\mathbb{P}$ satisfies *a temporal formula* φ *at time point* $k \geq 0$*, written* $\mathbf{M}, k \models \varphi$*, if the following conditions hold:*

1. $\mathbf{M}, k \models \top$ *and* $\mathbf{M}, k \not\models \bot$
2. $\mathbf{M}, k \models p$ *if* $p \in H_k$ *for any atom* $p \in \mathbb{P}$
3. $\mathbf{M}, k \models \varphi \wedge \psi$ *iff* $\mathbf{M}, k \models \varphi$ *and* $\mathbf{M}, k \models \psi$
4. $\mathbf{M}, k \models \varphi \vee \psi$ *iff* $\mathbf{M}, k \models \varphi$ *or* $\mathbf{M}, k \models \psi$
5. $\mathbf{M}, k \models \varphi \rightarrow \psi$ *iff* $\langle \mathbf{H}', \mathbf{T} \rangle, k \not\models \varphi$ *or* $\langle \mathbf{H}', \mathbf{T} \rangle, k \models \psi$*, for all* $\mathbf{H}' \in \{\mathbf{H}, \mathbf{T}\}$
6. $\mathbf{M}, k \models \circ\varphi$ *iff* $\mathbf{M}, k{+}1 \models \varphi$
7. $\mathbf{M}, k \models \varphi \mathbin{\mathbb{U}} \psi$ *iff for some* $j \geq k$*, we have* $\mathbf{M}, j \models \psi$ *and* $\mathbf{M}, i \models \varphi$ *for all* $k \leq i < j$
8. $\mathbf{M}, k \models \varphi \mathbin{\mathbb{R}} \psi$ *iff for all* $j \geq k$*, we have* $\mathbf{M}, j \models \psi$ *or* $\mathbf{M}, i \models \varphi$ *for some* $k \leq i < j$

$\square$

These conditions inherit the interpretation of connectives from LTL, with the exception of the implication which is treated as in the logic of HT. Implication must be satisfied in "both dimensions" $\mathbf{H}$ (here) and $\mathbf{T}$ (there) of the trace, using $\langle \mathbf{H}, \mathbf{T} \rangle$ (as in the other connectives) but also $\langle \mathbf{T}, \mathbf{T} \rangle$. An HT-trace $\mathbf{M}$ is a *model* of a temporal theory Γ if $\mathbf{M}, 0 \models \varphi$ for all $\varphi \in \Gamma$. As in the case of ITL, the *Temporal logic of Here-and-There* (THT for short) logic is induced by the set of all tautologies.

In [4] it is shown how THT semantics can be alternatively given as an extension of the $\mathrm{ITL}^{\mathrm{p}}$ logic, as stated in the following proposition.

Proposition 6 ([4]). $\mathrm{ITL}^{\mathrm{ht}} =$ *THT.*

As a consequence, all the results we can obtain in terms of HT persistent dynamic models can be equivalently applied to the case of HT-traces. In the remaining of this section, we provide the criterion for telling whether $\langle \mathbf{H}, \mathbf{T} \rangle$ is an equilibrium model of an input formula φ.

Definition 4 (Temporal Equilibrium/Stable Model). *A total HT-trace* $\langle \mathbf{T}, \mathbf{T} \rangle$ *is a* temporal equilibrium model *of a temporal formula* φ *if (1)* $\langle \mathbf{T}, \mathbf{T} \rangle, 0 \models \varphi$ *and (2) there is no HT-trace* $\langle \mathbf{H}, \mathbf{T} \rangle$ *such that* $\mathbf{H} < \mathbf{T}$ *and* $\langle \mathbf{H}, \mathbf{T} \rangle, 0 \models \varphi$*. In this case, the trace* $\mathbf{T}$ *is called a* temporal stable model *(*TS*-model) of* φ*.* $\square$

Temporal Equilibrium Logic (TEL) is the (non-monotonic) logic induced by temporal equilibrium models.

4 Two Fixpoint Characterisations of Propositional Equilibrium Logic

In this section, we present two well-known characterisations of equilibrium models (and, therefore, answer sets) that will be extended to the temporal case in Section 5. The first characterisation we are going to present was first defined in [20] and it is based on *theory completions*, a concept used in autoepistemic and default logic [17, 7].

Definition 5 ([20]). *Let Γ be a theory. A set E of formulas extending Γ is said to be a* completion *of Γ iff for every Boolean formula φ*

$$\Gamma \cup \{\neg\psi \mid \psi \notin E\} \models_{HT} \varphi \text{ iff } \varphi \in E.$$

Equilibrium models correspond precisely to completions in the Boolean case. For any model $\mathbf{M}$, we set $Th(\mathbf{M}) = \{\psi \mid \mathbf{M} \models_{\mathrm{HT}} \psi\}$. The relation between equilibrium models and theory completions is stated next.

Proposition 7 ([21]). *For any theory Γ, there is a one-to-one correspondence between the equilibrium models of Γ and the completions of Γ. In particular, a HT interpretation $\mathbf{M}$ is an equilibrium model of a theory Γ iff for all Boolean formulas φ,*

$$\Gamma \cup \{\neg\psi \mid \psi \notin Th(\mathbf{M})\} \models_{HT} \varphi \text{ iff } \varphi \in Th(\mathbf{M})$$

where $\models_{HT}$ stands for the semantic consequence operator in HT and $Th(\mathbf{M})$.

A slightly different fixpoint characterisation of equilibrium logic is called IL-*safe beliefs* [19], which are originally defined in terms of the semantic consequence operator in Intuitionistic Logic (IL), denoted by $\models_{\mathrm{IL}}$.

Definition 6. *A set of atoms T is said to be a IL-*safe belief *of a theory Γ if*

- *$\Gamma \cup \{\neg\neg p \mid p \in T\} \cup \{\neg p \mid p \notin T\}$ is IL-consistent and*
- *$\Gamma \cup \{\neg\neg p \mid p \in T\} \cup \{\neg p \mid p \notin T\} \models_{IL} T$*[1]*.*

In [19] it is also shown that IL can be replaced by any logic X satisfying IL $\subseteq$ X $\subseteq$ HT obtaining the same set of safe beliefs.

Proposition 8 ([19]). *Let T be a set of atoms and let X and Y be two intermediate logics satisfying $IL \subseteq X \subseteq Y \subseteq HT$. For any propositional theory Γ, T is a X-safe belief of Γ iff T is a Y-safe belief of Γ.*

[1] For simplicity, we admit here a set of atoms in the right part of the semantic consequence operator. By $\Gamma \models_{\mathrm{IL}} T$ we mean $\Gamma \models_{\mathrm{IL}} p$, for all $p \in T$.

When replacing IL by the strongest intermediate logic HT the following result can proven.

Corollary 1 ([19]). *For all propositional theory Γ and all set of atoms $T \subseteq \mathbb{P}$, T is a IL-safe belief of Γ iff T is a HT-safe belief of Γ.*

Finally, it was also shown that HT-safe beliefs correspond to equilibrium models, as stated in the following lemma

Lemma 1 ([20, 19]). *A set of atoms T is a HT-safe belief of a theory Γ iff $\langle T, T\rangle$ is an equilibrium model of Γ.*

Note that Proposition 8 and Lemma 1 show that we can replace HT by any intermediate logic and the set of equilibrium models remains the same. This is a strong argument to use the strongest intermediate logic HT as underlying monotonic logic of equilibrium logic.

5 Two Fixpoint Characterisations of Temporal Equilibrium Logic

In this section we will extend the fixpoint characterisations presented in Section 4 to the temporal case. In order to extend Pearce's characterisation, we need to reformulate some of his definitions. In this section, given a HT-trace, we *redefine* $Th(\mathbf{M}) = \{\varphi \mid \mathbf{M}, 0 \models_{\text{THT}} \varphi\}$.

Proposition 9. *Let $\mathbf{M} = \langle \mathbf{T}, \mathbf{T}\rangle$ be a temporal equilibrium model of Γ. For all HT-trace $\mathbf{M}' = \langle \mathbf{H}', \mathbf{T}'\rangle$, if $\mathbf{M}' \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\}$ then $\mathbf{T} = \mathbf{T}' = \mathbf{H}'$.*

Proof. Assume towards a contradiction that $\mathbf{M}', 0 \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\}$ but not $\mathbf{T} = \mathbf{T}' = \mathbf{H}'$. We first consider the case where $\mathbf{T} \neq \mathbf{T}'$. Then, there exists $i \geq 0$ such that $T_i \neq T'_i$. There is two cases:

- If $T_i \not\subseteq T'_i$, there exists $p \in T_i$ such that $p \notin T'_i$. Since $p \in T_i$, then $\mathbf{M}, 0 \models \circ^i p$. Since $\mathbf{M}$ is a total model, $\mathbf{M}, 0 \not\models \neg\circ^i p$. Therefore, $\neg\circ^i p \notin Th(\mathbf{M})$. Since $\mathbf{M}', 0 \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\}$, then $\mathbf{M}', 0 \models \neg\neg\circ^i p$. It follows that $\langle \mathbf{T}', \mathbf{T}'\rangle, 0 \models \circ^i p$ so $p \in T'_i$: a contradiction.
- If $T_i \not\supseteq T'_i$, there exists $p \in T'_i$ such that $p \notin T_i$. Since $p \notin T_i$ then $\mathbf{M}, 0 \not\models \circ^i p$. Therefore, $\circ^i p \notin Th(\mathbf{M})$. Since $\mathbf{M}', 0 \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\}$ then $\mathbf{M}', 0 \models \neg\circ^i p$. By the THT satisfaction relation it follows that $\langle \mathbf{T}', \mathbf{T}'\rangle, 0 \not\models \circ^i p$, so $p \notin T'_i$: a contradiction.

Therefore, we can assume that $\mathbf{T} = \mathbf{T}'$. For the second case $\mathbf{H}' \neq \mathbf{T}$, from $\mathbf{H}' \leq \mathbf{T}$ we have $\mathbf{H}' < \mathbf{T}$. Since $\langle \mathbf{T}, \mathbf{T}\rangle$ is a temporal equilibrium model of Γ and $\mathbf{H}' < \mathbf{T}$, then $\langle \mathbf{H}', \mathbf{T}\rangle \not\models \Gamma$, so $\mathbf{M}', 0 \not\models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\}$: a contradiction.

In the temporal case we can obtain the same result by replacing HT for THT as underlying logic, as stated in the following proposition.

Lemma 2. *For any theory Γ and any total HT-trace $\mathbf{M} = \langle \mathbf{T}, \mathbf{T} \rangle$, the following items are equivalent:*

(1) $\mathbf{M}$ *is a temporal equilibrium model of* Γ*;*
(2) for all $\varphi \in \mathcal{L}$, $\Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\} \models_{THT} \varphi$ *iff* $\varphi \in Th(\mathbf{M})$.

Proof. To prove that Item (1) implies Item (2) we assume that Item (1) holds but (2) does not. Then, $\mathbf{M}$ is a temporal equilibrium model of Γ but there exists a formula $\varphi \in \mathcal{L}$ such that either

- $\Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\} \models_{\mathrm{THT}} \varphi$ but $\varphi \notin Th(\mathbf{M})$: in this case, since $\mathbf{M}$ is a temporal equilibrium model of Γ then $\mathbf{M}, 0 \models \Gamma$. Moreover, since $\mathbf{M}$ is a total model we can easily check that $\mathbf{M}, 0 \models \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\}$. Therefore, $\mathbf{M}, 0 \models \varphi$ which contradicts $\varphi \notin Th(\mathbf{M})$, or
- $\varphi \in Th(\mathbf{M})$ but $\Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\} \not\models_{\mathrm{THT}} \varphi$: in this case, there there exists $\mathbf{M}' = \langle \mathbf{H}', \mathbf{T}' \rangle$ such that $\mathbf{M}', 0 \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\}$ but $\mathbf{M}', 0 \not\models \varphi$. From $\mathbf{M}', 0 \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\}$ and Proposition 9 it follows $\mathbf{M}' = \mathbf{M}$. Therefore, $\mathbf{M}, 0 \not\models \varphi$, which means that $\varphi \notin Th(\mathbf{M})$: a contradiction.

For the converse direction, let us assume towards a contradiction that $\mathbf{M}$ is not an equilibrium model of Γ. We have two possibilities:

- $\mathbf{M}, 0 \not\models \Gamma$. Therefore, $\Gamma \neq \emptyset$ and there exists $\varphi \in \Gamma$ such that $\mathbf{M}, 0 \not\models \varphi$. This means that $\varphi \notin Th(\mathbf{M})$. Since (2) holds, $\Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\} \not\models_{\mathrm{THT}} \varphi$. It follows that there exists $\mathbf{M}' = \langle \mathbf{H}', \mathbf{T}' \rangle$ such that $\mathbf{M}', 0 \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\}$ but $\mathbf{M}', 0 \not\models \varphi$. Since $\mathbf{M}', 0 \models \Gamma$ and $\varphi \in \Gamma$ then $\mathbf{M}, 0 \models \varphi$: a contradiction.
- $\mathbf{M}, 0 \models \Gamma$ but there exists $\mathbf{M}' = \langle \mathbf{H}', \mathbf{T} \rangle$ such that $\mathbf{H}' < \mathbf{T}$ and $\mathbf{M}', 0 \models \Gamma$. From $\mathbf{H}' < \mathbf{T}$ follows that there exists $i \geq 0$ and $\circ^i p \in \mathcal{L}$ such that $\mathbf{M}', 0 \not\models \circ^i p$, but $\mathbf{M}, 0 \models \circ^i p$. Since $\circ^i p \in Th(\mathbf{M})$ then $\Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\} \models_{\mathrm{THT}} \varphi$. The reader can be easily check that $\mathbf{M}', 0 \models \{\neg\varphi \mid \varphi \notin Th(\mathbf{M})\}$. Therefore, $\mathbf{M}', 0 \models \circ^i p$: a contradiction.

For extending Definition 6 to the temporal case, we need some extra definitions. Since safe beliefs correspond to sets of atoms and in the temporal case, the truth of an atom depends on the time point it is considered, we need to define the so-called set of temporal atoms associated to a signature $\mathbb{P}$ as follows: given a set of atoms $\mathbb{P}$, we define its associated set of *temporal atoms* as

$$\mathbb{P}^{\bigcirc} = \{\circ^i p \mid p \in \mathbb{P} \text{ and } i \geq 0\}.$$

Clearly, for $p \in \mathbb{P}$, $\circ^0 p = p$. We can readily now define the concept of *temporal safe belief*, which is also parametrised in terms of an intermediate temporal logic.

Definition 7. *Let Γ be a temporal theory. The set $S \subseteq \mathbb{P}^{\bigcirc}$ is said to be a* $\mathrm{ITL}^{\mathrm{p}}$*-temporal safe belief with respect to Γ if*

(1) $\Gamma \cup \{\neg\neg\circ^i p \mid \circ^i p \in S\} \cup \{\neg\circ^i p \mid \circ^i p \notin S\}$ *is* $\mathrm{ITL}^{\mathrm{p}}$*-consistent and*

(2) $\Gamma \cup \{\neg\neg\bigcirc^i p \mid \bigcirc^i p \in S\} \cup \{\neg\bigcirc^i p \mid \bigcirc^i p \notin S\} \models_{\text{ITL}^\text{p}} S$.

In Definition 7, ITL^p can be exchanged by any other intermediate temporal logic X satisfying $\text{ITL}^\text{p} \subseteq \text{X} \subseteq \text{THT}$. However for the sake of generality, we used the weakest logic we are considering (as it is the case in [19]). In order to provide the correspondence between temporal equilibrium models and temporal safe beliefs, let us consider the THT logic.

Definition 8. *Given a total HT-trace* $\langle \mathbf{T}, \mathbf{T}\rangle$ *we define* $S = \{\bigcirc^i p \mid p \in T_i\}$. *Clearly,* $S \subseteq \mathbb{P}^{\bigcirc}$. *Conversely, given* S *we define* $\langle \mathbf{T}, \mathbf{T}\rangle$ *as* $T_i = \{p \mid \bigcirc^i p \in S\}$.

Proposition 10. *For any temporal theory* Γ*, any total HT-trace* $\langle \mathbf{T}, \mathbf{T}\rangle$ *and set* $S \subseteq \mathbb{P}^{\bigcirc}$ *related as described in Definition 8, the following items are equivalent:*

(1) $\langle \mathbf{T}, \mathbf{T}\rangle$ *is a temporal equilibrium model of* Γ*;*
(2) S *is a THT-temporal safe belief of* Γ

Proof. To prove that (1) implies (2) let us assume that S is not a THT-temporal safe belief of Γ. Let us assume that $\Gamma \cup \{\neg\neg\bigcirc^i p \mid \bigcirc^i p \in S\} \cup \{\neg\bigcirc^i p \mid \bigcirc^i p \notin S\}$ is consistent but $\Gamma \cup \{\neg\neg\bigcirc^i p \mid \bigcirc^i p \in S\} \cup \{\neg\bigcirc^i p \mid \bigcirc^i p \notin S\} \not\models_{\text{THT}} S$. This means that there exists a HT trace $\mathbf{M} = \langle \mathbf{H}', \mathbf{T}'\rangle$ such that $\mathbf{M}, 0 \models \Gamma$, $\mathbf{M}, 0 \models \{\neg\neg\bigcirc^i p \mid \bigcirc^i p \in S\}$, $\mathbf{M}, 0 \models \{\neg\bigcirc^i p \mid \bigcirc^i p \notin S\}$ but $\mathbf{M}, 0 \not\models S$. From $\mathbf{M}, 0 \models \{\neg\neg\bigcirc^i p \mid \bigcirc^i p \in S\}$ and $\mathbf{M}, 0 \models \{\neg\bigcirc^i p \mid \bigcirc^i p \notin S\}$ we can conclude $\mathbf{T}' = \mathbf{T}$. From $\mathbf{M}, 0 \not\models S$ it follows that $\mathbf{M}, i \not\models p$, for some $i \geq 0$. From this and $\mathbf{M}, 0 \models \neg\neg\bigcirc^i p$ we get $\langle \mathbf{T}, \mathbf{T}\rangle, i \models p$ so $\mathbf{H}' < \mathbf{T}$. Since $\mathbf{M}, 0 \models \Gamma$ then $\langle \mathbf{T}, \mathbf{T}\rangle$ is not an equilibrium model of Γ: a contradiction.

Conversely, let us assume towards a contradiction that S is a THT-temporal safe belief of Γ but $\langle \mathbf{T}, \mathbf{T}\rangle$ is not a temporal equilibrium model of Γ. Since S is a THT-temporal safe belief of Γ then $\Gamma \cup \{\neg\neg\bigcirc^i p \mid \bigcirc^i p \in S\} \cup \{\neg\bigcirc^i p \mid \bigcirc^i p \notin S\}$ is consistent. Therefore, let $\mathbf{M} = \langle \mathbf{H}', \mathbf{T}'\rangle$ be a model of the latter theory. Since $\mathbf{M}, 0 \models \{\neg\neg\bigcirc^i p \mid \bigcirc^i p \in S\} \cup \{\neg\bigcirc^i p \mid \bigcirc^i p \notin S\}$ then $\mathbf{T}' = \mathbf{T}$. Since $\langle \mathbf{H}', \mathbf{T}\rangle, 0 \models \Gamma$ then $\langle \mathbf{T}, \mathbf{T}\rangle, 0 \models \Gamma$. Since $\langle \mathbf{T}, \mathbf{T}\rangle$ is not an equilibrium model of Γ there exists a HTtrace $\langle \mathbf{H}'', \mathbf{T}\rangle$ such that $\mathbf{H}'' < \mathbf{T}$ and $\langle \mathbf{H}'', \mathbf{T}\rangle, 0 \models \Gamma$. However, this contradicts Condition (2) of Definition 7.

Before finishing this section we would like to add a discussion about the extension of Proposition 8 to the temporal case. In Proposition 8 it is proved that, when restricted to the propositional case, the set of safe beliefs (and therefore the set of equilibrium models) is preserved no matter which (propositional) intermediate logic we choose. In the temporal case, it is easy to prove that every ITL^p-temporal safe belief is also a THT-temporal safe belief, as stated in the following proposition.

Proposition 11. *Let* X *and* Y *be two intermediate temporal logics satisfying* $\text{ITL}^\text{p} \subseteq X \subseteq Y \subseteq THT$. *Let us take a theory* Γ*. For any set of atoms* $S \subseteq \mathbb{P}^{\bigcirc}$*, if* S *is a* X*-temporal safe belief of* Γ *then* S *is a* Y*-temporal safe belief of* Γ*.*

Proof. If S is a set X-temporal safe belief of Γ then $\Gamma \cup \{\neg\neg\circ^i p \mid \circ^i p \in S\} \cup \{\neg\circ^i p \mid \circ^i p \notin S, i \geq 0\}$ is X-consistent. By Proposition 4, $\Gamma \cup \{\neg\neg\circ^i p \mid \circ^i p \in S\} \cup \{\neg\circ^i p \mid \circ^i p \notin S\}$ is Y-consistent. Assume now towards a contradiction that Condition (2) of Definition 7 does not hold in the logic Y. This means that there exists a model $\mathfrak{M}$ and a world w such that $\mathfrak{M}, w \models_{\mathrm{Y}} \Gamma \cup \{\neg\neg\circ^i p \mid \circ^i p \in S\} \cup \{\neg\circ^i p \mid \circ^i p \notin S, i \geq 0\}$ but $\mathfrak{M}, w \not\models_{\mathrm{Y}} S$. However, since $\mathrm{X} \subseteq \mathrm{Y}$, $\mathfrak{M}$ can be regarded under the prism of the logic X. Therefore, $\mathfrak{M}$ is also a witness that falsifies Condition (2) of Definition 7 within the logic X: a contradiction.

However, the converse direction, i.e. if S is a Y-temporal safe belief of Γ then S is also a X-temporal safe belief of Γ is left as future work. In the case of propositional intermediate logics, this direction is proved by weakening Γ under the presence of negated and double negated atoms. However, the transformations proposed in [19] must be extended to the temporal case and that is not clear for us how to do it.

6 Conclusions

In this paper we have revisited two well-known fixpoint characterisations of equilibrium logic and answer sets, which we have extended to the temporal case. These extensions justify (even more) the choice of temporal here-and-there as a monotonic basis of temporal equilibrium logic. This current work can be extended in several ways:

1. A converse of Proposition 11, which would allow us to determine that the X temporal safe beliefs of a theory Γ coincide, for any $\mathrm{ITL^p} \subseteq \mathrm{X} \subseteq \mathrm{THT}$.
2. Our definition of temporal safe beliefs is done in term of $\mathrm{ITL^p}$, whose decidability remains an open problem [5]. However, $\mathrm{ITL^p}$ preserves consistency as happens in the IL case. Other intuitionistic temporal logics (like $\mathrm{ITL^e}$) lack of this property [5]. As a future line of research we want to complete this picture by determining whether replacing THT by any intermediate logic extending $\mathrm{ITL^e}$we obtain the same temporal safe beliefs.
3. TEL has been extended with dynamic and timed operators [8, 6] whose fixpoint characterisations have been left for future work.

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