Contracted Temporal Equilibrium Logic

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Abstract

The stable model semantics of logic programs has been characterized by Equilibrium Logic, which is a non-monotonic formalism that selects models from the (monotonic) intermediate logic of Here-and-There. It provides stable models for arbitrary propositional formulas and has been fruitfully extended to different modal languages. Among them are theories in the syntax of Linear-Time Temporal Logic (LTL), giving rise to Temporal Equilibrium logic (TEL) based on Temporal Here-and-There (THT). In TEL, models are selected that minimize truth among THT traces of the same length. In this paper, we consider a selection that in addition may reduce the number of transitions in a trace, intuitively forming a contraction of it. We thus introduce *contracted* THT and *contracted* TEL on top of a model selection on a logical basis. The resulting c-stable models can be viewed as stable models in TEL that can not be summarized into a smaller trace. We illustrate contraction on several examples related to logic programming and explore several properties, like the relation to TEL and LTL, and in particular the connection to the LTL property of *stuttering*.

1 Introduction

(Linear-time) Temporal Equilibrium Logic (TEL) (Aguado et al. 2013) is a well-known extension of Equilibrium Logic (Pearce 2006), the nonmonotonic logic that characterizes answer sets of a logic program. Its semantics is defined by selecting certain models of a theory in Temporal Here-and-There (THT), a temporal extension of the intermediate logic of Here-and-There (Heyting 1930). These selected models have the form of traces that are said to be *in equilibrium* (also called *stable models* or *stable traces*) when a certain kind of minimality holds, obtaining in this way a non-monotonic entailment relation.

A TEL theory may have many stable models, such as the formula $\varphi = \Box(a \vee b)$, where each trace $\mathbf{T} = T_0 \cdot T_1 \cdot T_2 \cdots$ with T_i being either $\{a\}$ or $\{b\}$ is a stable model; in particular $\mathbf{T} = \{a\}\cdot\{a\}\cdot\{b\}\cdot\{b\}\cdot\{b\}\cdot\{a\}\cdot\{a\}\cdot\{a\}\cdots$ and $\mathbf{T}' = \{a\}\cdot\{b\}\cdot\{a\}$ are stable models. The is intuitively more succinct and may be preferred over $\mathbf{T}$. A natural question then is how to select stable models with relevant state transitions.

In the literature, trace selection by length has been considered; e.g., in (Schuppan and Biere 2005) the authors select models on the basis of the shortest counterexamples for model checking purposes. For planning problems, `ASP` solvers are usually run up to a certain plan horizon, aiming at the computation of shortest plans. Other approaches for selection over traces involve the use of minimization criteria with weighted atoms, see (Dodaro, Fionda, and Greco 2022) for LTL over finite traces among others.

However, rather than simply imposing a selection function on stable models, we are interested in providing, in the spirit of TEL, a semantics that selects models on a logical basis. The idea is that not only the truth of atoms is minimized, but in addition segments of a trace are summarized.

To illustrate this superficially on the formula from above, by contracting in the trace $\mathbf{T}$ the initial segment $\{a\}\cdot\{a\}$ into $\{a\}$, and similarly $\{b\}\cdot\{b\}\cdot\{b\}$ and $\{a\}\cdot\{a\}\cdot\{a\}\cdots$ into $\{b\}$ and $\{a\}$, respectively, we obtain $\mathbf{T}'$ which preserves φ under contraction, as b resp. a is true over the segment associated with each position. The trace $\mathbf{T}'$ is a model of φ that can not be contracted, and is thus selected. To see why taking the stable models of minimum length does not suffice, consider the following example.

Example 1 *Suppose that, to move to the airport from our office, we may go by bus or take a taxi. If we go by bus, we must make two bus stops, bs_1 and bs_2 before arriving whereas, if we go by taxi, we always have to stop at a crossroad c. The number of transitions we may take between two stops is not predetermined. The following* TEL *theory is one possible simplified formalization of this example (recall that $\Box$, $\Diamond$, $\circ$ stand for always, eventually, and next time, respectively):*

$$bus \vee taxi \tag{1}$$
$$\Box(bus \to \circ\Diamond bs_1) \tag{2}$$
$$\Box(bs_1 \to \circ\Diamond bs_2) \tag{3}$$
$$\Box(bs_2 \to \circ\Diamond airport) \tag{4}$$
$$\Box(taxi \to \circ\Diamond c) \tag{5}$$
$$\Box(c \to \circ\Diamond airport) \tag{6}$$

The stable models of (1)-(6) follow two different patterns:

1. $\{bus\} \cdot \emptyset^* \cdot \{bs_1\} \cdot \emptyset^* \cdot \{bs_2\} \cdot \emptyset^* \cdot \{airport\} \cdot \emptyset^*$
2. $\{taxi\} \cdot \emptyset^* \cdot \{c\} \cdot \emptyset^* \cdot \{airport\} \cdot \emptyset^*$

where, in both cases, we may replace the last $\emptyset^*$ by $\emptyset^\omega$, dealing with traces of infinite length. The shortest stable model corresponds to $\{taxi\} \cdot \{c\} \cdot \{airport\}$, where we take the taxi and it arrives in the fastest possible way, without any delay in each trip segment. We claim that the stable model

$\{bus\} \cdot \{bs_1\} \cdot \{bs_2\} \cdot \{airport\}$, although longer, should be incomparably minimal as well, as it corresponds to the shortest trace we may get *when we decide to take the bus*.

For developing contraction, we consider furthermore the following desiderata as a guidance: (D1) A contracted trace should be in equilibrium, that is, contraction selects from the stable traces. (D2) Consecutively repeated states, known as *stuttering*, should preferably be eliminated, if possible. (D3) Prevailing semantics such as LTL and TEL should be recoverable by including axioms into a theory.

Our main contributions are then as follows.

- We introduce contraction THT (cTHT), in which interpretations are structures $\langle \mathbf{T}', \mathbf{T}, \mu \rangle$ where μ maps segments of $\mathbf{T}$ to $\mathbf{T}'$, in a way such that the contracted (summarized) trace $\mathbf{T}'$ is sound with respect to inferences that could be made in the trace $\mathbf{T}$; that is, while inferences might be dropped, no new formulas are derivable in a summarized segment of $\mathbf{T}$. For the definition of entailment, we resort to a temporal version of the intermediate logic known as *Bounded Depth 2*.
- On top of contracted THT, we then define contracted TEL (cTEL) by model selection according to a preference relation. Intuitively, a trace is in equilibrium, if no proper summarization can be made. The resulting equilibrium models, called c-stable models, obey D1 because they are also regular stable models, and D2 for meaningful language fragments. For instance, for formulas without the next-operator ($\circ$) and without nested implication, we are able to prove that c-stable models coincide exactly with the regular stable models (D1) that are stutter-free (D2).
- Both LTL and TEL can be recovered from cTEL by adding suitable axioms; the well-known property of LTL that for $\circ$-free formulas states can be stuttered is then a corollary.
- We show that satisfiability (stable model existence) has in cTEL the same complexity as in TEL, which is EXPSPACE-complete (Bozzelli and Pearce 2015), and that for cTEL fragments, standard reasoning tasks can be modularly translated into TEL.

We believe that our work provides a basic framework for defining contraction and summarization of (stable) traces that can be utilized in various contexts, such as for generating example traces, condensing given traces, analyzing minimal plans, and many further applications.

2 Preliminaries

The syntax of THT (and TEL) is the same as for LTL. In particular, in this paper, we use the following notation. Given a (countable, possibly infinite) set $\mathcal{A}$ of propositional variables (called *alphabet*), *temporal formulas* φ are defined by the grammar:

$$\varphi ::= a \mid \top \mid \bot \mid \varphi_1 \otimes \varphi_2 \mid \circ\varphi \mid \varphi_1 \,\mathbf{U}\, \varphi_2 \mid \varphi_1 \,\mathbf{R}\, \varphi_2$$

where $a \in \mathcal{A}$ is an atom and $\otimes$ is any binary Boolean connective $\otimes \in \{\rightarrow, \wedge, \vee\}$. The last four cases correspond to the temporal connectives whose names are listed as follows: $\circ$ for *next*; $\mathbf{U}$ for *until*; and $\mathbf{R}$ for *release*. A formula φ is said to be $\otimes$*-free* if it does not contain any occurrence of some connective $\otimes$. We also define several common derived operators like the Boolean connectives $\neg\varphi =_{\text{def}} \varphi \rightarrow \bot$, $\varphi \leftrightarrow \psi =_{\text{def}} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, and the following temporal operators: *always* as $\Box\varphi =_{\text{def}} \bot \,\mathbf{R}\, \varphi$, *eventually* as $\Diamond\varphi =_{\text{def}} \top \,\mathbf{U}\, \varphi$, *final* as $\mathbf{F} =_{\text{def}} \neg\circ\top$, and *weak next* as $\widehat{\circ}\varphi =_{\text{def}} \circ\varphi \vee \mathbf{F}$. A *(temporal) theory* is a (possibly infinite) set of temporal formulas.

Although THT and LTL share the same syntax, they have different semantics, the former being a weaker logic than the latter. The semantics of THT relies on the concept of pairs of traces. In LTL, a *trace* $\mathbf{T}$ of length $\lambda \geq 1$ (possibly infinite, $\lambda = \omega$) is a sequence $\mathbf{T} = (T_i)_{[0..\lambda)}$ of sets $T_i \subseteq \mathcal{A}$. A THT-trace $\mathbf{M}$ is a pair $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ where $\mathbf{H}$ and $\mathbf{T}$ are LTL traces of the same length, $\mathbf{H} = H_0 \cdot H_1 \cdot \ldots$ and $\mathbf{T} = T_0 \cdot T_1 \cdot \ldots$, and we additionally require $H_i \subseteq T_i \subseteq \mathcal{A}$. We sometimes use the notation $|\mathbf{M}| =_{\text{def}} \lambda$ to stand for the length of the trace. We say that $\mathbf{T}$ is *infinite* if $|\mathbf{T}| = \omega$ and *finite* if $|\mathbf{T}| \in \mathbb{N}$. Given $a \in \mathbb{N}$ and $b \in \mathbb{N} \cup \{\omega\}$, we let $\big[a..b\big)$ stand for the set $\{i \in \mathbb{N} \mid a \leq i < b\}$ and, analogously, $\big(a..b\big]$ when $b \neq \omega$ stands for $\{i \in \mathbb{N} \mid a < i \leq b\}$.

Definition 1 (THT**-satisfaction)** *Let* $\mathbf{M}$ *be a* cTHT*-trace* $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ *over alphabet* $\mathcal{A}$ *and let* $\lambda = |\mathbf{H}| = |\mathbf{T}|$*. Then* $\mathbf{M}$ satisfies *a temporal formula* φ *at step* $k \in \big[0..\lambda\big)$*, written* $\mathbf{M}, k \models \varphi$*, if the following recursive conditions hold:*

1. $\mathbf{M}, k \models \top$ *and* $\mathbf{M}, k \not\models \bot$*;*
2. $\mathbf{M}, k \models p$ *if* $p \in H_k$ *for any atom* $p \in \mathcal{A}$*;*
3. $\mathbf{M}, k \models \varphi \wedge \psi$ *iff* $\mathbf{M}, k \models \varphi$ *and* $\mathbf{M}, k \models \psi$*;*
4. $\mathbf{M}, k \models \varphi \vee \psi$ *iff* $\mathbf{M}, k \models \varphi$ *or* $\mathbf{M}, k \models \psi$*;*
5. $\mathbf{M}, k \models \varphi \rightarrow \psi$ *iff* $\langle \mathbf{H}', \mathbf{T} \rangle, k \not\models \varphi$*, or* $\langle \mathbf{H}', \mathbf{T} \rangle, k \models \psi$ *for all* $\mathbf{H}' \in \{\mathbf{H}, \mathbf{T}\}$*;*
6. $\mathbf{M}, k \models \circ\varphi$ *iff* $k{+}1 < \lambda$ *and* $\mathbf{M}, k{+}1 \models \varphi$*;*
7. $\mathbf{M}, k \models \varphi \mathbf{U} \psi$ *iff for some* $j \in \big[k..\lambda\big)$*, we have* $\mathbf{M}, j \models \psi$ *and* $\mathbf{M}, i \models \varphi$ *for all* $i \in \big[k..j\big)$*;*
8. $\mathbf{M}, k \models \varphi \,\mathbf{R}\, \psi$ *iff for all* $j \in \big[k..\lambda\big)$*, we have* $\mathbf{M}, j \models \psi$ *or* $\mathbf{M}, i \models \varphi$ *for some* $i \in \big[k..j\big)$*.* ☐

If $\mathbf{H} = \mathbf{T}$ in Definition 1, then we obtain that $\langle \mathbf{T}, \mathbf{T} \rangle, 0 \models \varphi$ iff $\mathbf{T}$ is an LTL model of φ.

Definition 2 (TEL) *A total* cTHT*-trace* $\langle \mathbf{T}, \mathbf{T} \rangle$ *is a* temporal equilibrium model *(or* stable model*) of a theory* Γ *if* $\mathbf{T}, 0 \models \Gamma$ *and there is no* $\mathbf{H} \neq \mathbf{T}$ *such that* $\langle \mathbf{H}, \mathbf{T} \rangle \models \Gamma$*.*

3 Contracted THT

To compare two different traces, we start introducing the concept of a *contractor* function μ, a mapping that transforms indices $i \in \big[0..\lambda\big)$ from an interval of length λ into new positions $\mu(i)$ inside an interval of length $\lambda' \leq \lambda$.

Definition 3 (Contractor) *Let* $\lambda, \lambda' \in \mathbb{N} \cup \{\omega\}$ *be two trace lengths. A* contractor *function* μ *from* λ *to* λ' *is any surjective function of type* $\mu : \big[0..\lambda\big) \rightarrow \big[0..\lambda'\big)$ *that satisfies* $\mu(0) = 0$ *and is monotonic, i.e.,* $\mu(i+1) \leq \mu(i) + 1$ *for all* $i \in \big[0..\lambda\big)$ *and* $i + 1 < \lambda$*.* ☐

Note first that, by monotonicity, $\mu(i+1) \geq \mu(i)$ and so, $\mu(i+1)$ can only be either $\mu(i)$ or $\mu(i) + 1$. Second, as μ is surjective, all elements in $\big[0..\lambda'\big)$ have a preimage in $\big[0..\lambda\big)$,

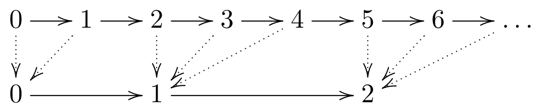

Figure 1: Example of contractor function μ_1.

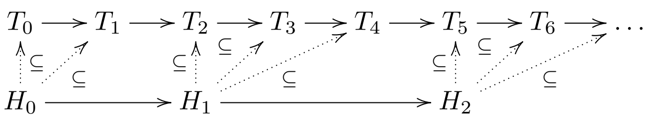

Figure 2: Subset relations imposed by $\mathbf{T} \downarrow^{\mu_1} \mathbf{H}$.

so $\lambda' \leq \lambda$, that is, μ generally produces an image interval of a smaller (or equal) length λ' than the original one λ (hence the name "contractor"). Moreover, for each point $i \in [0..\lambda')$ we can define its (non-empty) preimage set as usual:

$$\mu^{-}(i) =_{\text{def}} \{j \in [0..\lambda) \mid \mu(j) = i\}.$$

The identity function $id(i) =_{\text{def}} i$ is the only case of contractor in which the preimage $id^{-}(i)$ is a singleton – in other words, id^{-} constitutes an inverse *function*. An example of a contractor μ_1 for $\lambda = \omega$ and $\lambda' = 3$ is shown in Figure 1, where $\mu_1(i) = 2$ for all $i \geq 5$.

We can also see a contractor function μ as a way to organize the points in $[0..\lambda)$ into a sequence of λ' consecutive intervals so that $\mu^{-}(i)$ denotes the i-th interval (starting in 0). For instance, for μ_1 in Figure 1 we have the intervals ${\mu_1}^{-}(0) = [0..2)$, ${\mu_1}^{-}(1) = [2..5)$ and ${\mu_1}^{-}(2) = [5..\omega)$.

As we can see, we may have cases in which the contractor just leaves the same length $\lambda' = \lambda$, that is, there is no length contraction at all. If $\lambda \in \mathbb{N}$ this only happens with the identity function id: any other contractor will necessarily reduce the length $\lambda' < \lambda$. However, when $\lambda = \omega$ we have (infinitely many) other contractors μ different from id that do not reduce the interval length, leaving $\lambda' = \omega$ as well. As an example, we may take the function $\mu_2(i) =_{\text{def}} i \div 2$ for all $i \in \mathbb{N}$.

Definition 4 (Trace Contraction) *Let* $\mathbf{H}$ *and* $\mathbf{T}$ *be two traces of lengths* $\lambda_h = |\mathbf{H}|$ *and* $\lambda_t = |\mathbf{T}|$ *respectively. We say that a contractor* μ *from* λ_t *to* λ_h contracts $\mathbf{T}$ *to* $\mathbf{H}$*, written* $\mathbf{T} \downarrow^{\mu} \mathbf{H}$*, when* $T_i \supseteq H_{\mu(i)}$ *for all* $i \in [0..\lambda_t)$. $\square$

Definition 5 (Summarization $\preceq$) *We say that trace* $\mathbf{H}$ summarizes *trace* $\mathbf{T}$*, written* $\mathbf{H} \preceq \mathbf{T}$*, when there exists some contractor* μ *such that* $\mathbf{T} \downarrow^{\mu} \mathbf{H}$. $\square$

Figure 2 shows the inclusion relations (in dashed lines) imposed by $\mathbf{T} \downarrow^{\mu_1} \mathbf{H}$ using the contractor μ_1 in Figure 1 for two traces $\mathbf{T}$ of length $\lambda_t = \omega$ and $\mathbf{H}$ of length $\lambda_h = 3$. Informally speaking, when we jump from H_i to H_{i+1} we allow trace $\mathbf{T}$ to make any finite number of additional transitions, but all those new states must be supersets of H_i. In other words, all those new T_j are allowed to include more information than H_i, but never to remove any atom that is true at H_i. When $\mathbf{H}$ is shorter than $\mathbf{T}$, as in the example, the last state in $\mathbf{H}$, in the example H_2, must be a subset of *all* the remaining states in $\mathbf{T}$, even if the latter is infinite.

To introduce the relation with THT, we notice that:

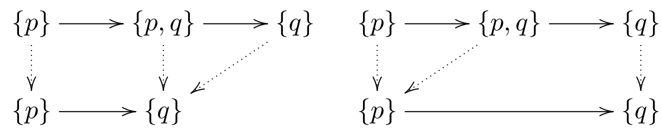

Figure 3: Two possible contractors to prove that $\mathbf{H} \prec \mathbf{T}$ for traces $\mathbf{H} = \{p\} \cdot \{q\}$ and $\mathbf{T} = \{p\} \cdot \{p, q\} \cdot \{q\}$.

Proposition 1 *The condition from Defn. 4 for* $\mathbf{T} \downarrow^{id} \mathbf{H}$ *amounts to:*

- $\lambda_h = \lambda_t$ *and, for all* $i \in [0..\lambda_h)$*,* $H_i \subseteq T_i$. $\square$

In fact, the condition we obtained above with $\mu = id$ amounts to the ordering relation among traces used in standard TEL (Aguado et al. 2021). Moreover, we will also use the previous notation $\mathbf{H} \leq \mathbf{T}$ to mean $\mathbf{T} \downarrow^{id} \mathbf{H}$. Note that $\mathbf{H} \leq \mathbf{T}$ implies $\mathbf{H} \preceq \mathbf{T}$ but the opposite is not true, for instance, $\preceq$ allows now comparing traces of different lengths.

Proposition 2 $\preceq$ *is a preorder relation among traces.*

In other words, $\preceq$ is reflexive and transitive but, in general, anti-symmetry may not hold (at least, for pairs of infinite traces). As a counterexample, consider the traces $\mathbf{T} = \emptyset \cdot \emptyset \cdot \{a\}^{\omega}$ and $\mathbf{T}' = \emptyset \cdot \{a\}^{\omega}$. We can observe that $\mathbf{T}' \downarrow^{id} \mathbf{T}$ whereas we can also contract $\mathbf{T} \downarrow^{\mu} \mathbf{T}'$ using the function $\mu(0) = 0$ and $\mu(i) = i - 1$ for all $i > 0$. This means we have both $\mathbf{T} \preceq \mathbf{T}'$ and $\mathbf{T}' \preceq \mathbf{T}$ but $\mathbf{T} \neq \mathbf{T}'$. However, in the finite case, anti-symmetry holds, and we thus have:

Proposition 3 $\preceq$ *is an order relation among finite traces.*

In the sequel, we write $\mathbf{H} \prec \mathbf{T}$ if $\mathbf{H} \preceq \mathbf{T}$ and $\mathbf{H} \neq \mathbf{T}$. It must be observed that, given two traces $\mathbf{H} \prec \mathbf{T}$, we may have more than one contractor function μ for which $\mathbf{T} \downarrow^{\mu} \mathbf{H}$. As a simple example, take $\mathbf{T} = \{p\} \cdot \{p, q\} \cdot \{q\}$ and $\mathbf{H} = \{p\} \cdot \{q\}$. To prove that these two traces satisfy $\mathbf{H} \prec \mathbf{T}$ we can use contractor μ with $\mu(1) = \mu(2) = 1$ or contractor μ' with $\mu'(1) = 0$ and $\mu'(2) = 2$ as shown in Figure 3.

Definition 6 (cTHT-trace) *A* cTHT*-trace for alphabet* $\mathcal{A}$ *is a triple* $\langle \mathbf{H}, \mathbf{T}, \mu \rangle$ *satisfying* $\mathbf{T} \downarrow^{\mu} \mathbf{H}$.

A cTHT-trace $\langle \mathbf{H}, \mathbf{T}, \mu \rangle$ is called *integral* when $\mu = id$ (there is no contraction) and *contracted* otherwise. An integral trace where we further have $\mathbf{H} = \mathbf{T}$ is said to be *total*.

Given a cTHT-trace $\langle \mathbf{H}, \mathbf{T}, \mu \rangle$, we call each $k \in [0..|\mathbf{H}|)$ a trace *step*. Moreover, step k is said to be *integral* when $\mu(k) = \{i\}$ is a singleton and when this happens, by abuse of notation, we may sometimes use $\mu^{-}(k)$ as a function denoting the element i. We say that step k is *contracted* when it is not integral, i.e., $|\mu^{-}(k)| > 1$. To put an example, for the contractor on the left of Figure 3, step 0 is integral because $\mu^{-}(0) = 0$ while step 1 is contracted as $\mu^{-}(1) = \{1, 2\}$. The opposite happens for the contractor on the right: in that case 0 is contracted $\mu^{-}(0) = \{0, 1\}$ and 1 is integral $\mu^{-}(1) = 2$.

We define next a particular kind of traces that we will consider later on.

Definition 7 (cTHT-satisfaction) *Let* $\mathbf{M}$ *be a* cTHT*-trace* $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu \rangle$ *over alphabet* $\mathcal{A}$ *and let* $\lambda = |\mathbf{H}|$*. Then* $\mathbf{M}$ satisfies *a temporal formula* φ *at step* $k \in [0..\lambda)$*, written*

$\mathbf{M}, k \models \varphi$, *if the same conditions 1–8 as in Defn. 1 hold excepting 5 and 6, which are modified as follows:*

5. $\mathbf{M}, k \models \varphi \to \psi$ *iff both:*
   (a) $\mathbf{M}, k \not\models \varphi$ *or* $\mathbf{M}, k \models \psi$;
   (b) $\langle \mathbf{T}, \mathbf{T}, id\rangle, j \not\models \varphi$ *or* $\langle \mathbf{T}, \mathbf{T}, id\rangle, j \models \psi \ \ \forall j \in \mu^-(k)$
6. $\mathbf{M}, k \models \circ\varphi$ *iff* $|\mu^-(k)| = 1$, $k{+}1 < \lambda$, *and* $\mathbf{M}, k{+}1 \models \varphi$ □

The intuitive meaning of the condition for the next operator $\circ\varphi$ is that the current step k must not be a final state $k{+}1 < \lambda$, it must be integral ($|\mu^-(k)| = 1$) and φ must hold at $k{+}1$. The fact that k is integral means that H_k cannot be an "abbreviation" of a sequence of $\mathbf{T}$ states above. That is, to satisfy $\circ\varphi$ at k we force the existence of a state at $k{+}1$ (as usual) but also that the transition from k to $k{+}1$ is integral for $\mathbf{T}$. To put an example, $\mathbf{M}, 0 \models \circ q$ in the interpretation $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu\rangle$ corresponding to the left diagram of Fig. 3, but not in the right as the transition from 0 to 1 is contracting in that case. The reason for this restriction has to do with persistence (anything satisfied in $\mathbf{H}$ must be satisfied in $\mathbf{T}$ too) as we will prove later on. Informally speaking, in the right diagram, from H_0 we can see that q holds at the next state H_1. But when we move above, we should check $\circ q$ both at T_0 and T_1 as $\mu^-(0) = \{0, 1\}$. Here this holds, but if T_1 were $\{p\}$ instead, then T_0 does not satisfy $\circ q$ and we would have a case where $\circ q$ is true in $\mathbf{H}$ but not in $\mathbf{T}$. If the transition is integral, as in the left diagram, we can guarantee that T_1 satisfies q because it is restricted by $H_1 = \{q\}$.

Proposition 4 *Satisfaction* $\langle \mathbf{H}, \mathbf{T}, id\rangle, k \models \varphi$ *is equivalent to satisfaction* $\langle \mathbf{H}, \mathbf{T}\rangle, k \models \varphi$ *in* THT.

Corollary 1 $\langle \mathbf{T}, \mathbf{T}, id\rangle, k \models \varphi$ *is equivalent to* $\mathbf{T}, k \models \varphi$ *in* LTL.

Due to these results, we may replace an integral trace $\langle \mathbf{H}, \mathbf{T}, id\rangle$ by the THT-trace $\langle \mathbf{H}, \mathbf{T}\rangle$ and $\langle \mathbf{T}, \mathbf{T}, id\rangle$ by $\mathbf{T}$ when using them in satisfaction relations. Given an interval or set S of time steps, we will also write $\mathbf{T}, S \models \varphi$ to stand for $\mathbf{T}, j \models \varphi$ for all $j \in S$. Using this result and notation, we can replace item 7(b) in Definition 7 by the simpler condition:

7(b′) $\mathbf{T}, \mu^-(k) \models \varphi \to \psi$.

The following result lifts an essential property of THT to the contracted setting: that every formula that is satisfied by a THT-trace $\langle \mathbf{H}, \mathbf{T}\rangle$ must be satisfied by $\mathbf{T}$ viewed as LTL-interpretation. It reflects the intuitionistic view that when moving from a state $\mathbf{H}$ to a state $\mathbf{T}$ with more truth information, inferences made will be preserved.

Theorem 1 (Persistence) *For every* cTHT-*trace* $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu\rangle$ *with* $\lambda = |\mathbf{H}|$ *and every* $k \in [0..\lambda)$*:* $\mathbf{M}, k \models \varphi$ *implies* $\mathbf{T}, \mu^-(k) \models \varphi$.

As in THT, the satisfaction of negation $\neg\varphi$ at point k amounts to an LTL check on the $\mathbf{T}$ component, but in this case, we must make that check on all the preimage points $\mu^-(k)$. Formally:

Proposition 5 *For every* cTHT-*trace* $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu\rangle$*, formula* φ*, and position* k*,* $\mathbf{M}, k \models \neg\varphi$ *iff* $\mathbf{T}, \mu^-(k) \models \neg\varphi$ *(in LTL).*

As usual, given a temporal formula φ for alphabet $\mathcal{A}$, we write $\models \varphi$ to represent that φ is a *tautology*, that is, $\mathbf{M}, k \models \varphi$ for every cTHT-trace $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu\rangle$ over $\mathcal{A}$ and every $k \in [0..|H|)$.

Definition 8 (entailment/equivalence) *Let* φ *and* ψ *be two temporal formulas over alphabet* $\mathcal{A}$*. We say that* φ entails ψ*, written* $\varphi \models \psi$*, when* $\mathbf{M}, k \models \varphi$ *implies* $\mathbf{M}, k \models \psi$*, for any trace* $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu\rangle$ *over* $\mathcal{A}$ *and every* $k \in [0..|\mathbf{H}|)$*. We say that* φ *and* ψ *are* equivalent*, written* $\varphi \equiv \psi$*, when both* $\varphi \models \psi$ *and* $\psi \models \varphi$*.*

Proposition 6 $\varphi \models \psi$ *iff* $\models \varphi \to \psi$.

Corollary 2 $\varphi \equiv \psi$ *iff* $\models \varphi \leftrightarrow \psi$.

A cTHT-trace $\mathbf{M}$ is a *model* of a theory Γ if $\mathbf{M}, 0 \models \varphi$ for all $\varphi \in \Gamma$. The following property from LTL and THT also holds in cTHT (yet, it is known to be false once we introduce past operators).

Proposition 7 $\varphi \equiv \psi$ *iff* φ *and* ψ *have the same models.*

Proposition 8 *The semantics induced for derived operators is the following. Let* $\mathbf{M}$ *be a* cTHT-*trace* $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu\rangle$ *over alphabet* $\mathcal{A}$ *and let* $\lambda = |\mathbf{H}|$.

1. $\mathbf{M}, k \models \Diamond\varphi$ *iff* $\mathbf{M}, j \models \varphi$ *for some* $j \in [k..\lambda)$
2. $\mathbf{M}, k \models \Box\varphi$ *iff* $\mathbf{M}, j \models \varphi$ *for all* $j \in [k..\lambda)$
3. $\mathbf{M}, k \models \mathsf{F}$ *iff* $|\mu^-(k)| = 1$ *and* $k{+}1 = \lambda$
4. $\mathbf{M}, k \models \widehat{\circ}\varphi$ *iff* $|\mu^-(k)| = 1$ *and either* $k{+}1 = \lambda$ *or* $\mathbf{M}, k{+}1 \models \varphi$.

It is well-known that THT is a strictly weaker logic than LTL (Aguado et al. 2021), that is THT $\subset$ LTL. Proposition 4 allows proving that cTHT $\subseteq$ THT, namely, that any cTHT-tautology is also an THT-tautology. We may also observe that this relation is strict, cTHT $\subset$ THT. For instance, while $\widehat{\circ}\top \equiv \circ\top \vee \neg\circ\top$ is a tautology in LTL and in THT, it is not a tautology any more in cTHT. Indeed, from Proposition 8.4 we conclude that $\mathbf{M}, k \models \widehat{\circ}\top$ iff $|\mu^-(k)| = 1$, that is, satisfying $\widehat{\circ}\top$ at point k just means requiring that k is an integral transition. Thus, we can take any interpretation where k is contracting, such as $k = 0$ in Figure 2, to falsify $\widehat{\circ}\top$. Furthermore, including the axiom:

$$\Box\widehat{\circ}\top \qquad \textbf{(INT)}$$

forces all steps to be integral (no contraction), and so, $\mu = id$ collapsing to THT. In other words, cTHT+ **(INT)** = THT.

Similarly, we may also observe that the THT-equivalent formulas $\circ\top$ and $\neg\neg\circ\top (= \neg\mathsf{F})$ are not equivalent in cTHT either. While satisfying $\circ\top$ at k asserts that k is integral and jumps to a state $k{+}1$, $\neg\mathsf{F}$ just means the preimage of k does not contain the last position in $\mathbf{T}$.

Proposition 9 *Let* $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu\rangle$ *be a* cTHT-*trace. Then,* $\mathbf{M}, k \models \neg\mathsf{F}$ *iff* $\max(\mu^-(k)) + 1 < |\mathbf{T}|$.

One important observation is that the non-temporal fragment of cTHT is actually weaker than HT. If we restrict to propositional connectives $\vee, \wedge, \to, \bot, \top$ and atoms, the satisfaction relation collapses to an intermediate logic whose Kripke models have the form of "forks", namely, one point (or world) H_0 that can see a group of worlds T_i for $i \in \mu^-(0)$

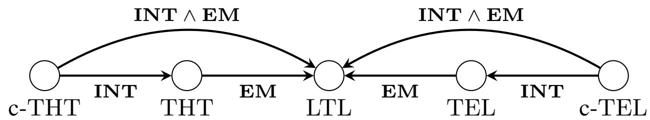

Figure 4: Relations between LTL, THT, and cTHT.

with no (intuitionistic) accessibility among them. This structure corresponds to the intermediate logic of Bounded Depth 2 (BD2), one of the seven interpolable intermediate logics (Maksimova 1977), like HT itself, although BD2 is also strictly weaker. For instance, BD2 does not satisfy the principle of *weak excluded middle* $\neg\varphi \vee \neg\neg\varphi$, which is an HT-tautology. In some sense, cTHT can be seen as one of the (possible) temporal extensions of BD2.

We have seen that axiom (**INT**) allows collapsing cTHT into THT. It is also known that THT collapses to LTL by the addition of the *(temporal) excluded middle axiom* (**EM**) scheme. We also introduce its weaker version (**WEM**):

$$\mathbf{EM} := \Box(p \vee \neg p) \tag{7}$$

$$\mathbf{WEM} := \Box(\neg\neg p \vee \neg p) \tag{8}$$

for every $p \in \mathcal{A}$. Figure 4 depicts some reductions among different logics obtained by the inclusion of axioms.

Although cTHT is strictly weaker than LTL, there are syntactic fragments on which LTL-equivalences are still applicable for cTHT. For instance:

Proposition 10 *Let φ, ψ be a pair of $\rightarrow$-free, $\circ$-free formulas. Then, the formula $\varphi \leftrightarrow \psi$ is a* cTHT*-tautology iff it is an* LTL*-tautology.*

As an illustration, the LTL-tautology

$$p \,\mathbf{U}\, \Diamond q \quad \leftrightarrow \quad \Diamond q \tag{9}$$

is also a cTHT-tautology because the formulas on the two sides of the double implication are $\rightarrow$-free and $\circ$-free. Note that we can still exploit this result to prove properties about formulas with $\circ$ or $\rightarrow$. To put an example, the formula

$$\circ\varphi \,\mathbf{U}\, \Diamond(\psi \rightarrow \gamma) \quad \leftrightarrow \quad \Diamond(\psi \rightarrow \gamma) \tag{10}$$

is still a cTHT-tautology because cTHT satisfies the law of uniform substitution, whose validity can be proved by contradiction, and we can replace p by $\circ\varphi$ and q by $(\psi \rightarrow \gamma)$ in (9) to obtain (10).

4 Contracted Temporal Equilibrium Logic

We are now introducing a selection criterion over cTHT models, which requires the nonexistence of a proper logical summarization. As pointed out in (Lamport 1983), standard temporal logics like LTL cannot express *possibilities* (or the absence of possibilities) over different behaviors; for that, a second-order logic is needed. TEL already provides a notion of possibility, in the sense that a selection criteria over models is employed, but its purpose is to simulate the stable semantics on a temporal setting only.

Definition 9 (cTEL) *A total* cTHT*-trace $\langle \mathbf{T}, \mathbf{T}, id\rangle$ is a* contracted temporal equilibrium model *(or* c-stable model*) of a theory Γ if it is a model of Γ (that is $\mathbf{T}, 0 \models \Gamma$ in* LTL*) and there is no model $\langle \mathbf{H}, \mathbf{T}, \mu\rangle$ of Γ with $\mathbf{H} \prec \mathbf{T}$.*

If we constrain μ in Definition 9 to identity id, then we obtain the definition of (integral) temporal equilibrium model as in (Aguado et al. 2013) for infinite and finite traces, which we call *stable models*. A first observation is then the following.

Theorem 2 *Any c-stable model of a theory Γ is also a (standard) stable model of Γ.*

Hence, the desidered condition (D1) is satisfied. To see how the semantics works, let us see some examples.

Example 2 *Consider the theory $\Gamma = \{\Diamond p\}$. Its stable models are the traces $\mathbf{T}$ with p at a single state, i.e., $\mathbf{T} = \emptyset^m \cdot \{p\} \cdot \emptyset^n$ with $m, n \geq 0$ and, possibly, $n = \omega$. Let us try to build an $\mathbf{H}$ and μ such that $\langle \mathbf{H}, \mathbf{T}, \mu\rangle \models \Diamond p$ and $\mathbf{H} \prec \mathbf{T}$.*

- *Consider $\mathbf{T} = \{p\}$. Then $|\mathbf{H}| = 1$ and $\mu = id$ must hold; as $\mathbf{T}$ is stable, $\mathbf{H} \prec \mathbf{T}$ is not possible, so $\mathbf{T}$ is c-stable.*
- *Consider $\mathbf{T} = \emptyset \cdot \{p\}$. Then $H_0 = \{p\}$ is not possible as $\mu(0) = 0$, which forces $H_0 \subseteq T_0 = \emptyset$. So, we must have $H_0 = \emptyset$; as $\langle \mathbf{H}, \mathbf{T}, \mu\rangle, 0 \models \Diamond p$ must hold, this forces $H_1 = \{p\}$, $\mu(1) = 1$ and so $\mathbf{H} = \mathbf{T}$. Again $\mathbf{T}$ is c-stable.*
- *Also trace $\mathbf{T} = \{p\} \cdot \emptyset$ is c-stable: $|\mathbf{H}| = 1$ would force $p \in H_0$ and $H_0 \subseteq T_1 = \emptyset$, while $|\mathbf{H}| = 1$ forces $\mu = id$; then $H_1 \subseteq T_1 = \emptyset$ and so $H_0 = \{p\}$ but then $\mathbf{H} = \mathbf{T}$.*
- *The trace $\mathbf{T} = \emptyset \cdot \{p\} \cdot \emptyset$ is c-stable: $|\mathbf{H}| = 3$ would force $\mu = id$, and as $\mathbf{T}$ is stable, $\mathbf{H} \prec \mathbf{T}$ is not possible. Otherwise, $\mu(2) = 1$ would force $H_1 = \emptyset$ thus $H_0 = \{p\} \subseteq T_0 = \emptyset$; $\mu(2) = 0$ would force $\mathbf{H} = H_0 = \emptyset$ but then $\langle \mathbf{H}, \mathbf{T}, \mu\rangle, 0 \models \Diamond p$ is not possible.*
- *Take any trace $\mathbf{T} = \emptyset^m \cdot \{p\} \cdot \emptyset^n$ where $m > 1$. Then, we can build $\mathbf{H} = \emptyset \cdot \{p\} \cdot \emptyset^n$ and use $\mu(i) = m + i - 1$ for all $i \geq 1$ for the model $\mathbf{H} \prec \mathbf{T}$, so $\mathbf{T}$ is not c-stable.*
- *Finally for any trace $\mathbf{T} = \emptyset^m \cdot \{p\} \cdot \emptyset^n$ with $n > 1$ we similarly build $\mathbf{H} = \emptyset^m \cdot \{p\} \cdot \emptyset$ and use $\mu(i) = i$ for all $i \in [0..m)$ (remember the last state forces $\mu(m+1) = \omega$) for the desired model $\mathbf{H} \prec \mathbf{T}$, so $\mathbf{T}$ is not c-stable.*

In conclusion, this theory has only four *c-stable models: $\{p\}$, $\{p\} \cdot \emptyset$, $\emptyset \cdot \{p\}$, and $\emptyset \cdot \{p\} \cdot \emptyset$, which are compactly represented with the regular expression as $\mathbf{T} = \emptyset^? \cdot \{p\} \cdot \emptyset^?$.* $\Box$

Example 3 *Consider the dual theory $\Gamma = \{\,\Box p\,\}$. Its stable models are $\mathbf{T} = \{p\}^\lambda$ where $\lambda \geq 1$ and possibly $\lambda = \omega$. The only c-stable stable model is $\mathbf{T} = \{p\}$: for any $\lambda > 1$ we can use $\mathbf{H} = \{p\}$ and readily show that $\mathbf{T}$ is not c-stable.*

Example 4 *Consider next the theory Γ with the formulas*

$$\Diamond p \tag{11}$$

$$\Box(p \rightarrow \circ p \vee q). \tag{12}$$

To satisfy (11)*, any stable model $\mathbf{T}$ must make p true at some point k, and by* (12) *p must be true forever, i.e., $T_i = \{p\}$ for $i \in [k..\lambda)$, or until both p and q are true at some point $k' \geq q$, i.e., $T_i = \{p\}$ for $i \in [k..k')$ and $T_{k'} = \{p, q\}$. By the minimality condition of $\mathbf{T}$, no p or q can appear before k or after k', i.e., $T_i = \emptyset$ for $i \in [0..k)$ and $i \in [k'{+}1..\lambda)$, as we could make them all false in a smaller $\mathbf{H}$ for an* HT*-model; likewise, we could make all p at $i \in [k..\lambda)$ resp. $i \in [k..k')$ false. Thus, $\mathbf{T}$ must be, written as regular expression, of the form $\mathbf{T} = \emptyset^* \cdot \{p, q\} \cdot (\emptyset^* + \emptyset^\omega)$.*

It is not hard to check that any nonempty subsequence $\emptyset^k$ in $\mathbf{T}$, including $\emptyset^\omega$, can be contracted into $\emptyset$, leading to four c-stable models: $\{p,q\}$, $\emptyset \cdot \{p,q\}$, $\{p,q\} \cdot \emptyset$, and $\emptyset \cdot \{p,q\} \cdot \emptyset$, which are compactly represented as $\mathbf{T} = \emptyset^? \cdot \{p,q\} \cdot \emptyset^?$. $\square$

Example 5 *Let then $\Gamma = \{ \Diamond(p \wedge \Box(p \rightarrow \circ p \vee q)) \}$. In LTL, this theory is weaker than Γ in Example 4, which we rename to Γ', as (12) is nested into (11); thus we have $\Gamma' \models_{\mathrm{LTL}} \Gamma$; the same holds in THT and in cTHT.*

Consequently, any stable model $\mathbf{T}$ of Γ must be a stable model of Γ'. Conversely, any stable model $\mathbf{T}$ of Γ' is an LTL model of Γ, and by its particular form, we can not form a smaller $\mathbf{H}$ such that $\langle \mathbf{H}, \mathbf{T} \rangle \models \Gamma$; hence $\mathbf{T}$ is also a stable model of Γ. This likewise holds for c-stable models. Thus Γ and Γ' have the same stable resp. c-stable models. $\square$

We next consider occurrence of negation.

Example 6 *Let $\Gamma = \{\Box(\neg p \vee p)\}$. As THT plus $\mathbf{EM}$ collapses to LTL, any trace is a stable model of Γ that intuitively represents a choice for p or $\neg p$ at each point.*

To see which of them are c-stable, whenever we have a state repetition in $\mathbf{T}$, i.e., $T_i = T_{i+1}$ for some $i \geq 0$, then we can contract $\mathbf{T}$ to $\mathbf{T}'$ by leaving out T_{i+1}, i.e., we let $\mu(j) = j$ for $j \in [0..i]$ and $\mu(j) = j-1$ for $j \in [i..|\mathbf{T}|)$, and obtain $\langle \mathbf{T}', \mathbf{T}.\mu \rangle \models \Gamma$; if all T_j, $j > i$, are the same, we can also set $\mu(j) = i$. As this can be repeated, the c-stable models of Γ are all traces that alternate between $\{p\}$ and $\emptyset$; formally, they are captured by the regular expression

$$\emptyset\cdot(\{p\}\cdot\emptyset)^*\cdot\{p\}^? + \{p\}\cdot(\emptyset\cdot\{p\})^*\cdot\emptyset^? + (\emptyset\cdot\{p\})^\omega + (\{p\}\cdot\emptyset)^\omega$$

$\square$

In the examples above, repeated states have been eliminated, as desired by condition (D2). Clearly, this is not always possible.

Example 7 *Take $\Gamma = \{p, \circ p\}$. Its stable models are $\mathbf{T} = \{p\} \cdot \{p\}(\emptyset^* + \emptyset^\omega)$, and the formula $\circ p$ forces any contraction μ to be integral at step 0; the c-stable models are $\{p\} \cdot \{p\}$ and $\{p\} \cdot \{p\} \cdot \emptyset$.*

The $\circ$-operator can be seen as a way to state that a given *transition cannot be contracted*. Thus, when a $\circ$-formula is derived, we may have repetitions of states that cannot be removed in c-stable models; as we shall see in the next section, we can safely remove stuttering when we deal with $\circ$-free formulas, so (D2) will be satisfied.

Regarding (D3), we readily obtain from the discussion about LTL, THT and cTHT in the previous section that $c\mathrm{TEL}+(\mathbf{INT}) = \mathrm{TEL}$ and $\mathrm{TEL}+(\mathbf{EM}) = \mathrm{LTL}$, completing the diagram in Figure 4.

5 Characterising c-Stable Models

We notice that c-stable models are ω-regular languages, which follows from an automata construction for deciding the satisfiability problem in cTEL. Intuitively, this can be already seen from Defn. 9, as we need to produce an LTL automaton for the *guess* $\mathbf{T}$, and a THT automaton for producing a *defeater* $\mathbf{H}$ and μ. After completing the defeater automaton, we project away all the atoms referring to $\mathbf{H}$ and μ, and we compute the intersection of the two automata. All the above mentioned automata operators are closed under ω-regular languages (Büchi 1960).

We further note that for $\circ$-free formulas, an alphabet of size at least 2 is needed for having an aperiodic c-stable trace. For instance, the trace $\mathbf{T} = \emptyset \cdot a \cdot \emptyset \cdot ab \cdot \emptyset \cdot (ab)^2 \cdot \emptyset \cdot (ab)^3 \cdot \ldots$ is a c-stable trace for the formula $\Box((a \vee \neg a) \wedge (b \vee \neg b))$.

5.1 $\circ$-free formulas

We now turn our attention to $\circ$-free formulas, i.e., formulas without the $\circ$-operator. The absence of the intricate semantic behavior of the latter allows us to identify sufficient conditions for the existence of c-stable models as well as characterizations for classes of $\circ$-free theories, and under restricted contractions for all such theories.

A key notion for this endeavor is stuttering of traces.

Definition 10 (Stuttering) *A trace $\mathbf{T}$ is a* stuttering *of a trace $\mathbf{T}'$ if $\mathbf{T} \downarrow^\mu \mathbf{T}'$ for some μ such that $T_i = T'_{\mu(i)}$, for all $i \in [0..|\mathbf{T}|)$; it is* proper *if, in addition, $\mathbf{T} \neq \mathbf{T}'$.*

That is, in a stuttering the same state is repeated, possibly multiple times or even infinitely often; properness ensures that $\mathbf{T}$ must have some repetition that is not in $\mathbf{T}'$.

Let us consider what happens when we "pump" LTL and THT models of a set Γ of $\circ$-free formulas.

Lemma 1 (Stutter Equivalences) *Suppose $\mathbf{T}$ (resp. $\mathbf{H}$) is a stuttering of $\mathbf{T}'$ (resp. $\mathbf{H}'$) via contraction μ. If φ is $\circ$-free, then for each $j \in [0, \lambda')$,*

1. *$\mathbf{T}', j \models \varphi$ iff $\mathbf{T}, \mu^{-1}(j) \models \varphi$, and*
2. *$\langle \mathbf{H}', \mathbf{T}' \rangle, j \models \varphi$ iff $\langle \mathbf{H}, \mathbf{T} \rangle, \mu^{-1}(j) \models \varphi$.*

Item 1 of Lemma 1 is a well-known result of the LTL $\circ$-free fragment, while 2 is an immediate generalization of 1. Notably, we can summarize stuttered intervals in the There-trace using contractors while preserving THT satisfaction:

Proposition 11 (T-stutter Equivalence) *Let $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu \rangle$ where $T_i = T_j$ for every $i, j \in \mu^{-}(k)$ and $k \in [0, |\mathbf{T}'|)$, and let $\mathbf{H} \downarrow^\mu \mathbf{T}'$ be a stuttering of $\mathbf{T}'$. If φ is a $\circ$-free formula, then for each $k \in [0, |\mathbf{T}|)$, we have $\mathbf{M}, \mu(k) \models \varphi$ iff $\langle \mathbf{H}, \mathbf{T} \rangle, k \models \varphi$.*

Proposition 11 allows us to link summarisation inference to ordinary HT-inference. Armed with this proposition, we then show the following property. Let us call a trace $\mathbf{T}$ *stutter-free*, if it is not a proper stuttering of any sequence $\mathbf{T}'$.

Proposition 12 *For any set Γ of $\circ$-free formulas, $\mathbf{T}$ is a c-stable model of Γ only if $\mathbf{T}$ is a stable model of Γ such that $T_i \neq T_{i+1}$ for all $i \in [0, \lambda)$, i.e., $\mathbf{T}$ is stutter-free.*

Thus, stutter-freeness is a necessary condition for c-stability in the absence of the $\circ$-operator. On the other hand this condition is not sufficient in general, as shown by the following example.

Example 8 *Consider the theory $\Gamma = \{\neg p \rightarrow \Diamond p\}$, which has the stable model $\mathbf{T} = \emptyset \cdot \{p\}$: at $i = 0$, p is false and thus p must be true at $i = 1$. However, while $\mathbf{T}$ is stutter-free, it is not a c-stable model. Indeed, for $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu \rangle$ where $T' = \emptyset$ and $\mu(0) = \mu(1) = 0$, we have $\mathbf{M} \models \Gamma$ as $\mathbf{M} \not\models \neg p$ and $\mathbf{T} \models \Diamond p$. Intuitively, contraction of $\mathbf{T}$ into $\mathbf{T}'$ affects stability as the antecedent $\neg p$ is no longer provable.* $\square$

To obtain a fragment of TEL for which stutter-freeness is a sufficient condition for summarization, we thus have to impose some restrictions. This intuitively regards negation respectively implication, as summarization does not affect provability of formulas without implication, which we call *positive formulas*. As it turns out, by excluding nested implication we achieve this goal.

Let THT^1 denote the class of all formulas without nested implication. We then obtain the following result, which informally generalizes the only-if direction of Proposition 11 for THT^1 theories.

Proposition 13 *Let $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu \rangle$ and $\mathbf{H} \downarrow^{\mu} \mathbf{T}'$ be a stuttering of $\mathbf{T}'$. If φ is a $\circ$-free formula from THT^1, then for each $k \in [0, \lambda')$, $\mathbf{M}, k \models \varphi$ implies $\langle \mathbf{H}, \mathbf{T} \rangle, \mu^-(k) \models \varphi$.*

The converse direction does not hold, as shown by $\mathbf{T}' = \emptyset$, $\mathbf{T} = \{p\}\cdot\emptyset$, μ as obvious, $\mathbf{H} = \emptyset\cdot\emptyset$, and $\Gamma = \{\Diamond(p \vee \neg p)\}$.

From Proposition 13, we obtain the converse of Proposition 12 for THT^1:

Proposition 14 *Suppose Γ is a set of $\circ$-free formulas from THT^1. Then $\mathbf{T}$ is a c-stable model of Γ if $\mathbf{T}$ is a stable model of Γ such that $T_i \neq T_{i+1}$ for all $i \in [0, \lambda)$, i.e., $\mathbf{T}$ is not a proper stuttering of any sequence $\mathbf{T}'$.*

From Propositions 12 and 14, we then obtain the characterization of the c-stable models in terms of stable models.

Theorem 3 *For any $\circ$-free THT^1 theory Γ, (i) the c-stable models coincide with the stutter-free stable models; (ii) a c-stable model exists iff a stable model exists; and (iii) every stable model becomes c-stable by removing all repetitions.*

Revisiting Example 6, we see that the c-stable models of Γ are captured by the characterization in Theorem 3.

Even $\circ$-free THT^1 formulas allows us to enforce infinite models sensitive to c-stability.

Example 9 *By adding the formula $\Box(\circ\top)$ to the theory $\Gamma = \{\Box(\neg p \vee p)\}$ in Example 6, as well as to any theory, all models of Γ must be infinite, and only the two infinite traces*

$$(\emptyset \cdot \{p\})^{\omega} \text{ and } (\{p\} \cdot \emptyset)^{\omega})$$

would remain as c-stable models.

Infinite models may be also enforced by adding the formula

$$\Box((q \vee \neg q) \wedge \Diamond q \wedge \Diamond \neg q) \tag{13}$$

where is q is an auxiliary atom. While $\Box(\circ\top)$ restricts any mapping $\mathbf{T} \downarrow^{\mu} \mathbf{T}'$ between traces to identity ($\mu = id$) and thus c-stability falls back to stability, formula (13) preserves the full mappability.

We note that Theorem 3 can be extended to non-THT^1 theories by imposing syntactic conditions. In particular, it continues to hold if Γ contains for each non-THT^1 formula φ in Γ also $(\mathbf{EM})$ for each variable p that occurs in φ; this enforces totality of models on the variables $\mathcal{A}_{\varphi}$ in φ, and thus the condition in Proposition 11 applies relative to $\mathcal{A}_{\varphi}$. As LTL equals $\mathrm{THT} + (\mathbf{EM})$ (cf. Figure 4), the stutter-free LTL-models of any $\circ$-free theory Γ are thus characterized by the c-stable models of $\Gamma \cup (\mathbf{EM})$, i.e., by the contractions of the stable models of Γ under classical semantics.

We further remark that $(\mathbf{EM})$ also belong to the syntactic fragment THT^1 whereas its inclusion captures full LTL: we can convert each formula φ in LTL into negation normal form in polynomial , by rewriting implication to disjunction and moving negations inside formulas – in particular, using $\neg(\varphi \,\mathsf{U}\, \psi) \equiv \neg\varphi \,\mathsf{R}\, \neg\psi$ and $\neg(\varphi \,\mathsf{R}\, \psi) \equiv \neg\varphi \,\mathsf{U}\, \neg\psi$ in LTL, where double negation cancels.

5.2 GP-formulas

While the class THT^1 excludes nested implication, it is still expressive and allows for encoding EXPSPACE-complete problems, even if $\Diamond$ and $\Box$ are the only temporal operators available (Bozzelli and Pearce 2015). In particular, it embraces theories that include statements of the form

$$\alpha_1 \wedge \cdots \wedge \alpha_m \to \beta_1 \vee \cdots \vee \beta_n, \tag{14}$$
$$\Box(\alpha_1 \wedge \cdots \wedge \alpha_m \to \beta_1 \vee \cdots \vee \beta_n) \tag{15}$$
$$\Box(\Box\alpha_1 \to \beta_1) \tag{16}$$
$$\Box(\alpha_1 \to \Diamond\beta_1) \tag{17}$$
$$\alpha_1 \vee \neg\alpha_1 \tag{18}$$

$m, n \geq 0$, where all α_i and β_j are positive formulas. They are rules of temporal logic programs (Aguado et al. 2021) if the formulas are atoms in (16)-(17) and atoms or formulas $\circ p$ in (14)-(15) where p is an atom. For $m = 0$ these are (disjunctive) facts, and for $n = 0$ constraints where the consequent is $\bot$). The formula (18) is a *guessing rule* which makes in stable semantics α_1 either true or false, which then leads to two different scenarios reflected in different stable models (so they exist).

It is well-known that for *positive disjunctive logic programs*, which are sets of formulas (14) where all α_i and β_j are atoms, the stable models coincide with the $\subseteq$-minimal (in short, minimal) models. We now present the class GP (standing for *Generalized Positive*) of formulas with an analogous property for theories Γ over this class. The c-stable models of $\circ$-free Γ theories are then the minimal stutter-free models of Γ; furthermore, for no different c-stable models $\mathbf{T}'$ and $\mathbf{T}$ of Γ, we can have $\mathbf{T}' \prec \mathbf{T}$ (which for arbitrary $\circ$-free theories is possible, cf. Example 6).

Definition 11 (GP formulas) *The class GP consists of all THT^1 formulas φ where each subformula $\varphi_1 \vee \varphi_2$, $\varphi_1 \,\mathsf{U}\, \varphi_2$, or $\varphi_1 \,\mathsf{R}\, \varphi_2$ of φ is positive unless $\varphi_1 = \bot$.*

Clearly GP properly generalizes positive formulas; e.g. $p \to q$ is positive and an admissible GP formula, but $\Box(p \wedge \Diamond r \to q \vee \Box s)$ is only admissible GP. The restriction on U and R subformulas mirrors the restriction on disjunction $\vee$, since both operators involve temporal disjunction. i.e., over time instants. Exempting the case $\varphi_1 = \bot$ means that $\varphi_1 \vee \varphi_2$ amounts to φ_2, which then simply must be from THT^1, $\varphi_1 \,\mathsf{U}\, \varphi_2$ amounts similarly to φ_2, and $\varphi_1 \,\mathsf{R}\, \varphi_2$ amounts to $\Box\varphi_2$, which intuitively is a temporal conjunction of φ_2 over the timeline.

Notably, positive (negation-free) temporal logic programs with rules of the form (14)–(17) fall into the class GP, and several examples considered above involve GP formulas.

Furthermore, with $\Box\circ\top$ but also with $\circ$-free GP formulas we may enforce infinite LTL models, such as with $\Box((p \to$

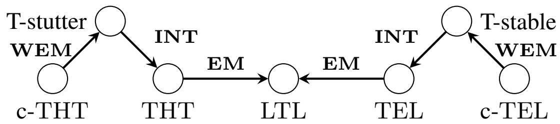

Figure 5: T-stuttering and T-stable semantics vs. the other semantics.

$\Diamond q) \wedge (q \rightarrow \Diamond p))$; by further adding $\Box((p \vee q) \wedge (p \wedge q \rightarrow \bot))$, each infinite stable trace gives rise to some c-stable model.

Based on Theorem 3, we then obtain the following characterization of c-stable models.

Theorem 4 *For any theory Γ of $\circ$-free* GP *formulas, the following conditions are equivalent:*

(1) $\mathbf{T}$ *is a c-stable model of* Γ*;*

(2) $\mathbf{T}$ *is a stutter-free $\subseteq$-minimal model of* Γ*, i.e., $T_i \neq T_{i+1}$ for all $i \in [0, \lambda)$;*

(3) $\mathbf{T}$ *is a ($\subseteq$-minimal) model of* Γ *and no model $\mathbf{T}' \prec \mathbf{T}$ of Γ exists.*

We remark that Theorem 4 does not hold for THT^1 formulas, which is easily witnessed by $\Gamma = \{p \vee \neg p\}$ and $\mathbf{T} = \{p\}$: $\mathbf{T}$ is a c-stable model but the conditions (2) and (3) are not satisfied. As this example shows, requiring in (2) and (3) that $\mathbf{T}$ is an stable model would not change this. However, the conditions (2) and (3) are sufficient for c-stability.

Proposition 15 *For any $\circ$-free theory Γ of THT^1 formulas, $\mathbf{T}$ is a c-stable model of Γ if (1) $\mathbf{T}$ is a stutter-free $\subseteq$-minimal model of Γ, or (2) $\mathbf{T}$ is a model of Γ and no model $\mathbf{T}' \prec \mathbf{T}$ of Γ exists.*

5.3 Taming summarization

As we illustrated with Example 8, if our theory is not in the fragment THT^1, stutter-free stable models may not c-stable, even in the absence of the next-operator. An intuitive explanation for this is that arbitrary contractions can be aggressive and compromise stability of formulas, as negation has to be evaluated over a segment in the There-trace. Specifically, this happens in Example 8 for the contraction of the stable model $\mathbf{T} = \emptyset \cdot \{p\}$ of $\Gamma = \{\neg p \rightarrow \Diamond p, \neg p \rightarrow \circ p\}$ to $\mathbf{T}' = \emptyset$, where the antecedent $\neg p$ of the implications has to be evaluated over $\emptyset \cdot \{p\}$, while for stability, it is only evaluated over $T_0 = \emptyset$.

Similarly, aggressive summarization may eliminate stable models if a change of axioms should only affect local stability, as for the theories $\Gamma_1 = \{\Box(p \vee q)\}$ and $\Gamma_2 = \{\Box(\neg p \rightarrow q), \Box(\neg q \rightarrow p)\}$, which have the same stable models. However, while the c-stable models of Γ_1 are its (infinitely many) stutter-free stable models, which are all the finite and infinite traces $\mathbf{T}$ that alternate between $\{p\}$ and $\{q\}$, Γ_2 has only $\{p\}$ and $\{q\}$ as c-stable models: each different stable model $\mathbf{T}$ can be contracted by μ to $\mathbf{T}' = \emptyset$, for which $\langle \mathbf{T}', \mathbf{T}, \mu \rangle \models \Gamma_2$ holds.

Aggressive summarization can be avoided by restricting contractions. In particular, the elimination of repetitions, which is a necessary feature of c-stable models, would be sufficient. To this end, we introduce the following notion.

Definition 12 (T-stutter,T-stable model) *We call a* cTHT-*trace $\langle \mathbf{H}, \mathbf{T}, \mu \rangle$* T-stutter, *if $T_i = T_j$ whenever $\mu(i) = \mu(j)$, for every $i, j \in [0..|\mathbf{T}|)$. Furthermore, a trace $\mathbf{T}$ is a* T-stable model *of a theory Γ if $\mathbf{T} \models \Gamma$ and no T-stutter $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu \rangle$ exists such that $\mathbf{M} \models \Gamma$ and $\mathbf{T}' \neq \mathbf{T}$.*

The less aggressive summarization allows us to recover all the $\circ$-free THT tautologies.

Proposition 16 *Let φ, ψ be a pair of $\circ$-free formulas. Then, $\varphi \leftrightarrow \psi$ is a T-stuttering tautology iff it is a* LTL*-tautology.*

In Example 8, we thus can't contract $\mathbf{T} = \emptyset \cdot \{p\}$ to $\mathbf{T}' = \emptyset$, and $\mathbf{T}$ is T-stable; similarly, no stable model $\mathbf{T}$ of Γ_2 with alternating $\{p\}$ and $\{q\}$ can be contracted to any trace $\mathbf{T}' \neq \mathbf{T}$ such that $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu \rangle$ is T-stutter and $\mathbf{M} \models \Gamma_2$. Clearly,

Proposition 17 *Every c-stable model $\mathbf{T}$ of a theory Γ is a T-stable model of Γ.*

It is not hard to see that we can constrain models $\mathbf{M}$ to be T-stutter by the temporal weak excluded middle axiom. We thus obtain the following characterization.

Theorem 5 *For any theory Γ, the T-stable models of Γ coincide with the c-stable models of $\Gamma \cup (\mathbf{WEM})$.*

The refined picture of semantics is shown in Figure 5. Many of the results for c-stable models of THT^1 theories similarly hold for T-stable models. The main result is an analogon of Theorem 3 for all $\circ$-free theories.

Theorem 6 *For any $\circ$-free theory Γ, (i) the T-stable models coincide with the stutter-free stable models; (ii) some T-stable model exists iff some stable model exists; and (iii) every stable model becomes T-stable by removing all repetitions.*

In other words, the T-stable models capture the stutter-free stable models of any $\circ$-free theory. Furthermore,

Corollary 3 *For any $\circ$-free THT^1 theory Γ, the T-stable models of Γ coincide with the c-stable models of Γ.*

As a final remark, when we consider the non-temporal fragment of cTHT under T-stutter traces, it is easy to see that we return back to the HT intermediate logic – although the current world in the BD2 structure may still see multiple accessible worlds in $\mathbf{T}$, all of them have the same valuation of atoms and collapse to a single point.

6 Computational Complexity and Reasoning

The satisfiability problem for LTL is well-known to be PSPACE-complete, while Bozzelli and Pearce (2015) showed that for TEL it is EXPSPACE-complete. They gave a detailed complexity picture for syntactic fragments $\mathrm{THT}^n_m(Op_1, \ldots, Op_k)$, where n is the implication depth, m the temporal operator depth, and (optional) the $Op_1, \ldots, Op_k$ are the admitted temporal operators, focusing on infinite traces but with corollaries for arbitrary and finite traces. We present the following novel results ('c' stands for 'complete').

Theorem 7 (TEL **complexity**) *Deciding* TEL *satisfiability of a theory, i.e., whether it has some stable model, from*

- $\mathrm{GP}_2(\Box, \Diamond)$ *is EXPSPACE-c for infinite traces;*
- $\mathrm{THT}^2_2(\mathsf{U})$ *is EXPSPACE-c for finite and for infinite traces.*
- $\mathrm{THT}^1_2(\circ, \Box, \Diamond)$ *is PSPACE-hard for finite traces, and* $\mathrm{THT}^1_2(\circ, \Box, \Diamond, \widehat{\circ}, \mathsf{F})$ *is in PSPACE for finite traces.*

- $\mathrm{THT}_2^1(\Box, \Diamond, \mathbf{F})$ *is NP-c for finite traces.*

Notably, as a corollary we get that deciding TEL satisfiability of $\neg$-free (and $\circ$-free) temporal programs from (Cabalar and Schaub 2019) is PSPACE-c (is NP-c), while it remains EXPSPACE-c in the general case.

For c-stable semantics, we easily obtain from our discussion about enforcing infinite traces (see Section 5) and the results in (Bozzelli and Pearce 2015) some lower bounds, as we can block non-id mappings by adding the formula $\Box\circ\top$.

Lemma 2 *Deciding whether a given theory* Γ *has some c-stable model is*

1. *EXPSPACE-hard in general and for* $\mathrm{THT}_{m+1}^1(\Diamond, \Box)$, $\mathrm{GP}_2(\Diamond, \Box, \circ)$*, and* $\mathrm{THT}_2^2(\mathbf{U})$ *and*
2. *PSPACE-hard for* $\mathrm{THT}^1(\circ, \mathbf{R})$ *and* THT^0.

The general case has a matching upper bound, which can be shown using automata-theoretic techniques, similar as for the result that satisfiability for TEL is in EXPSPACE. Hence, cTEL does not have higher complexity than TEL.

Theorem 8 (cTEL-complexity) *Deciding* cTEL *satisfiability of a theory* Γ*, i.e., whether* Γ *has a c-stable model, is EXPSPACE-complete.*

Combining our results in Section 5 with results in (Bozzelli and Pearce 2015), we obtain some upper bounds for infinite models (remind the non-duality between $\mathbf{U}$ and $\mathbf{R}$ in TEL):

Proposition 18 *Let* $\mathbf{INF} =_{\mathrm{def}} \Box\Diamond\circ\top$ *denote an operator expressing infinity. Then deciding whether a given theory* Γ *has an infinite c-stable model is*

1. *in PSPACE for* $\mathrm{THT}^1(\mathbf{R})$ *and* $\mathrm{THT}^1(\mathbf{U}, \mathbf{INF})$*, and*
2. *in* Σ_2^p *for* $\mathrm{THT}(\Diamond, \mathbf{INF})$.

Notably, in many cases, the existence of a stable model for a formula φ is, as shown by (Bozzelli and Pearce 2015), equivalent to the existence of a stable model $\mathbf{T}$ that is *strongly ultimately periodic*, i.e., that some $j \geq k$ exists such that $T_i = T_j$ for all $i \geq j$, where j is bounded in the size of the formula φ. For c-stability, the ultimate periodic part may either remain or be reduced to a finite suffix of the trace; thus, finite c-stable model existence is covered as well.

A detailed study of the complexity of cTHT and cTEL is beyond this paper. We remark, however, that are also low complexity fragments of cTEL semantics. In particular,

Proposition 19 *Deciding whether a given* $\circ$*-free theory* Γ *has some c-stable model is NP-complete for* THT_1^1 *and* GP_1.

This holds as some c-stable model exists in case of THT_1^1 (which includes GP_1) by Theorem 3 iff some stable model exists iff some infinite stable model exists (as states can be stuttered), which is NP-complete to decide (Bozzelli and Pearce 2015). The NP-hardness holds for GP_1, as it includes positive disjunctive logic programs (empty rule heads permitted), for which deciding stable model existence is well-known to be NP-complete (Eiter and Gottlob 1995). Proposition 19 thus shows that the benign complexity of a major class of logic programs extends to a meaningful temporal analogue.

6.1 Reasoning

Theorem 6 can be exploited to obtain the T-stable models of Γ from the stable models of an extension of Γ.

Let *diff* be a fresh atom and let Γ_{diff} be defined as

$$\begin{aligned}\Gamma_{\mathit{diff}} = \Gamma \;\cup\; \{\, \mathit{diff},\, \Box(\neg \mathit{diff} \to \bot)\} \cup \\ \{\Box((p \wedge \circ\neg p) \vee (\neg p \wedge \circ p) \to \circ \mathit{diff}) \mid p \in \mathcal{A}\}.\end{aligned}$$

That is, *diff* must be derived at each position; it is a fact at position 0, but at later positions can be derived iff adjacent positions are different. In case $\mathcal{A}$ or Γ is finite, we can also eliminate the auxiliary atom. Formally, we obtain:

Proposition 20 *The T-stable models* $\mathbf{T}$ *of any theory* Γ *of* $\circ$*-free* THT^1 *formulas correspond 1-1 to the stable models* $\mathbf{T}'$ *of* Γ_{diff}*, where* $T'_i = T_i \cup \{\mathit{diff}\}$ *for all* $i \in [0, \lambda)$.

From Theorem 6, We obtain that inference from the T-stable models of Γ is captured by inference from the stable models of Γ_{diff}. Furthermore, by Corollary 1, for $\circ$-free formulas, inference from the stable models is preserved:

Theorem 9 *Let* Γ *be a* $\circ$*-free theory and* φ *be a formula over* $\mathcal{A}$*. Then the following conditions are equivalent:*

(1) $\mathbf{T} \models \varphi$ *for every / some T-stable model* $\mathbf{T}$ *of* Γ*;*
(2) $\mathbf{T} \models \varphi$ *for every / some stable model* $\mathbf{T}$ *of* Γ_{diff}*;*
(3) $\mathbf{T} \models \varphi$ *for every / some stable model* $\mathbf{T}$ *of* Γ*, if* φ *is* $\circ$*-free.*

Since by Corollary 3 the T-stable models of $\circ$-free THT^1 theories coincide with the c-stable models, we obtain for this syntactic fragment an analogous result to Theorem 9 with with c-stable models in place of T-stable models.

7 Related Work and Conclusion

As mentioned in the Introduction, other works such as (Schuppan and Biere 2005; Dodaro, Fionda, and Greco 2022) aim at cost-based trace selection in LTL, using length or weighted atoms. Our setting relies on a non-monotonic logic, and contraction may preserve patterns that are eliminated by cost based selection, e.g. for $\Gamma = \{\,\Box(p \vee \neg p)\}$.

Stutter-invariance of $\circ$-free formulas in LTL is widely used, and our results generalize it to cTEL thanks to the recoverage of LTL. Extensions to nesting of $\circ$ and patterns in LTL have been studied, cf. (Kucera and Strejcek 2002), which would be interesting to explore for cTEL as well.

In planning, some approaches render simplified plans, such as CEGAR planning, (Seipp and Helmert 2013) or hierarchical planning, where macro-actions are composed of concrete actions. However, both are different from summarization in c-stable models: the former focuses on sets of states whereas the latter has an I/O flavor, disregarding intermediate states.

Another related line is the HyperLTL extensions of LTL using sets of traces for modeling concurrent processes. This was recently enriched with control of moving/stuttering traces (Baumeister et al. 2021) and lockstepwise traversal of subtraces removing "redundant" positions HyperLTL (Bozzelli, Peron, and Sánchez 2021). While our contraction establishes some asynchronous relationship between traces, it aims to support non-monotonic inference rather than to control execution traces; possible connections remain for study.

Outlook Our core work can be continued in several directions. Regarding logic and semantics, the connection to modal logics may be further investigated, as well as logical properties such as normal forms or equivalence in cTHT and cTEL. Further characterizations of the c-stable models, in particular in the presence of the next-operator, are an intriguing issue.

For computation, refining the complexity picture is suggestive and will help in guiding the development of suitable algorithms and implementations, especially for finite traces, where existing solvers such as telingo may be used.

Finally, we also plan to explore the application of c-stable models in the context of planning and explanation finding, such as for constructing plans or counterexamples.

Ethical Statement

There are no ethical issues.

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Appendix. Proofs.

Contracted THT

Given a contractor from λ' to λ, the following lemma shows that if j is the last position in $\left[0..\lambda'\right)$ then $\mu(j)$ is the last position in $\left[0..\lambda\right)$.

Lemma 3 *Let μ be a contractor from λ' to λ. If j+1 $= \lambda'$ then $\mu(j)$+1 $= \lambda$.*

Proof. Let $k = \mu(j)$. Suppose j+1 $= \lambda'$ but k+1 $< \lambda$. As μ is surjective, we should have some $i \in \mu^-(k\text{+}1)$, $i \in \left[0..\lambda'\right) = \left[0..j\text{+}1\right)$. Thus, we conclude $i < j$+1, that is $i \leq j$ and then k+1 $= \mu(i) \leq \mu(j) = k$ (by monotonicity), reaching a contradiction. □

Similarly, the next lemma shows that if k is the last position in $\left[0..\lambda\right)$ and the preimage of k is a singleton $\{j\}$, then j is the last position in $\left[0..\lambda'\right)$.

Lemma 4 *Let μ be a contractor from λ' to λ. If k+1 $= \lambda$ and $\mu^-(k) = \{j\}$ then j+1 $= \lambda'$.*

Proof. Suppose k+1 $= \lambda$ but j+1 $< \lambda'$. Then, we can apply the contractor on j+1 to get $\mu(j\text{+}1) \in \left[0..\lambda\right) = \left[0..k\text{+}1\right)$ and so $\mu(j\text{+}1) < k$+1 that is $\mu(j\text{+}1) \leq k$. By monotonicity $\mu(j\text{+}1) \geq \mu(j) = k$ so we conclude $\mu(j\text{+}1) = k$ which contradicts $\mu^-(k) = \{j\}$. □ We next observe that contractors can be composed.

Lemma 5 *If μ is a contractor from λ to λ' and μ' is a contractor from λ' to λ'' then $\nu = \mu' \cdot \mu$ is a contractor from λ to λ''.*

Proof. Note first that $\nu : \left[0..\lambda\right) \to \left[0..\lambda''\right)$ since μ maps from $\left[0..\lambda\right)$ to $\left[0..\lambda'\right)$ and this is mapped, in its turn, to $\left[0..\lambda''\right)$ by μ'. We prove next that ν is surjective. Since μ' is surjective, for all $i \in \left[0..\lambda''\right)$ we have $\mu'(j) = i$ for some $j \in \left[0..\lambda'\right)$, but as μ is surjective too, we must also have $j = \mu(k)$ for some $k \in \left[0..\lambda\right)$. But then $i = \mu'(j) = \mu'(\mu(k)) = \nu(k)$. Finally, we must prove $\nu(i) \leq \nu(i+1) \leq \nu(i) + 1$ for all i such that $i + 1 \in \left[0..\lambda\right)$. Since $\mu(i) \leq \mu(i+1) \leq \mu(i) + 1$ we have two cases

1. If $\mu(i+1) = \mu(i)$, then $\nu(i+1) = \mu'(\mu(i+1)) = \mu'(\mu(i)) = \nu(i)$.
2. If $\mu(i+1) = \mu(i) + 1$, then let us call $j = \mu(i)$ so that:

   $$\nu(i+1) = \mu'(\mu(i+1)) = \mu'(\mu(i)+1) = \mu'(j+1) \quad (19)$$

   Again, we have two cases: either $\mu'(j+1) = \mu'(j)$ or $\mu'(j\text{+}1) = \mu'(j)\text{+}1$. If $\mu'(j\text{+}1) = \mu'(j)$ then $\nu(i\text{+}1) = \mu'(j+1) = \mu'(j) = \mu'(\mu(i)) = \nu(i)$. In the second case, $\mu'(j+1) = \mu'(j) + 1$ and then $\nu(i+1) = \mu'(j+1) = \mu'(j) + 1 = \mu'(\mu(i)) + 1 = \nu(i) + 1$.

□

Proof of Proposition 2. We prove reflexivity and transitivity.

1. *Reflexivity*: to prove $\mathbf{T} \preceq \mathbf{T}$ we just take $\mu = id$, so by Proposition 1, the condition for $\mathbf{T} \downarrow^{id} \mathbf{T}$ amounts to $T_i \subseteq T_i$ that is tautological.
2. *Transitivity*: if $\mathbf{H} \preceq \mathbf{T}$ and $\mathbf{T} \preceq \mathbf{T}'$ there exist contractors μ and μ' such that $\mathbf{T} \downarrow^{\mu} \mathbf{H}$ and $\mathbf{T}' \downarrow^{\mu'} \mathbf{T}$, respectively. We define the function $\nu = \mu' \cdot \mu$ that is, the consecutive application of μ and μ'. We will prove that $\mathbf{T}' \downarrow^{\nu} \mathbf{H}$. Let $\lambda_h = |\mathbf{H}|$, $\lambda_t = |\mathbf{T}|$ and $\lambda'_t = |\mathbf{T}'|$. By Lemma 5, ν is a contractor from λ'_t to λ_h. Then, we must prove $H_{\nu(i)} \subseteq T'_i$ for all $i \in \left[0..\lambda'_t\right)$. Take some $i \in \left[0..\lambda'_t\right)$. From $\mathbf{T}' \downarrow^{\mu'} \mathbf{T}$ we know that $T_{\mu'(i)} \subseteq T'_i$. Let $j = \mu'(i)$: from $\mathbf{T} \downarrow^{\mu} \mathbf{H}$ we also know $H_{\mu(j)} \subseteq T_j$ that is $H_{\nu(i)} = H_{\mu(\mu'(i))} \subseteq T_{\mu'(i)} \subseteq T'_i$.

□

Proof of Proposition 3. Given Proposition 2, we remain to prove that, for finite traces, the $\preceq$ relation is antisymmetric. Let $\mathbf{T}$, $\mathbf{T}'$ be two finite traces such that $\mathbf{T} \preceq \mathbf{T}'$ and $\mathbf{T}' \preceq \mathbf{T}$, that is, there exist contractors μ and μ' such that $\mathbf{T} \downarrow^{\mu} \mathbf{T}'$ and $\mathbf{T}' \downarrow^{\mu'} \mathbf{T}$. Since μ and μ' are two surjective functions mapping two finite sets back and forth, then they are also injective, that is, both μ and μ' are bijective. This also implies both traces have the same length $|\mathbf{T}| = |\mathbf{T}'| = \lambda$. Now, since μ is monotonic, this implies $\mu(0) \leq \mu(1) \leq \cdots \leq \mu(\lambda - 1)$ and as μ is bijective, the only possibility is $\mu(i) = i$ for all $i \in \left[0..\lambda\right)$. Thus, $\mu = id$ and an analogous reasoning applies to conclude $\mu' = id$. Finally, when $\mu = \mu' = id$ the conditions $\mathbf{T} \downarrow^{\mu} \mathbf{T}'$ and $\mathbf{T}' \downarrow^{\mu'} \mathbf{T}$ imply $T_i \subseteq T'_i$ and $T'_i \subseteq T_i$ for all $i \in \left[0..\lambda\right)$, that is $\mathbf{T} = \mathbf{T}'$. □

Proof of Theorem 1 (Persistence). We proceed by structural induction.

- If $\varphi = \bot$ or $\varphi = \top$ the result follows trivially
- If $\varphi = p$ then $\mathbf{M}, k \models p$ amounts to $p \in H_k$. From $\mathbf{T} \downarrow^{\mu} \mathbf{H}$, we have $H_k \subseteq T_j$ for $j \in \mu^-(k)$ so we conclude $p \in T_j$ for all those j and the latter is equivalent to $\mathbf{T}, j \models p$ for j in that range.
- If $\varphi = \alpha \vee \beta$ then $\mathbf{M}, k \models \varphi$ is equivalent to the disjunction of $\mathbf{M}, k \models \alpha$ or $\mathbf{M}, k \models \beta$. We can apply the induction hypothesis on the subformulas to conclude either (1) $\mathbf{T}, \mu^-(k) \models \alpha$ or (2) $\mathbf{T}, \mu^-(k) \models \beta$. But clearly, any of the two cases imply $\mathbf{T}, \mu^-(k) \models \alpha \vee \beta$.
- If $\varphi = \alpha \wedge \beta$ the proof is analogous to the previous step $\varphi = \alpha \vee \beta$.
- If $\varphi = \alpha \to \beta$ and $\mathbf{M}, k \models \alpha \to \beta$ we conclude both
  - $\mathbf{M}, k \not\models \alpha$ or $\mathbf{M}, k \models \beta$;
  - $\mathbf{T}, \mu^-(k) \models \alpha \to \beta$;

  and the second item is directly what we need to prove.
- If $\varphi = \circ\alpha$ then $\mathbf{M}, k \models \varphi$ implies $|\mu^-(k)| = 1$, k+1 $< \lambda$ and $\mathbf{M}, k\text{+}1 \models \alpha$. Let us call i to the only element in $\mu^-(k) = \{i\}$. From k+1 $< \lambda$ we conclude there is a next state in $\mathbf{T}$, and so, i+1 $\in \mu^-(k\text{+}1)$. By induction hypothesis on $\mathbf{M}, k\text{+}1 \models \alpha$ we obtain $\mathbf{T}, \mu^-(k\text{+}1) \models \alpha$ and, in particular $\mathbf{T}, i\text{+}1 \models \alpha$. Finally, the latter is equivalent to $\mathbf{T}, i \models \circ\alpha$ and so $\mathbf{T}, \mu^-(k) \models \alpha$ since i is the only element in that set.
- If $\varphi = \alpha \mathbin{\mathbf{U}} \beta$, then $\mathbf{M}, k \models \alpha \mathbin{\mathbf{U}} \beta$ implies there exists $j \in \left[k..\lambda\right)$ such that $\mathbf{M}, j \models \beta$ and $\mathbf{M}, i \models \alpha$ for all

$i \in [k..j)$. We can apply the induction hypothesis on $\mathbf{M}, j \models \beta$ to conclude $\mathbf{T}, \mu^-(j) \models \beta$. Now, we consider two cases: (i) $j = k$ or (ii) $j > k$. For case (i), we may simply note that $\mathbf{T}, \mu^-(j) \models \beta$ amounts to $\mathbf{T}, \mu^-(k) \models \beta$ and so, $\mathbf{T}, \mu^-(k) \models \alpha \mathbf{U} \beta$ follows trivially, since in LTL, any state satisfying β trivially satisfies $\alpha \mathbf{U} \beta$. For case (ii), we have $j > k$. Let us define $m = \max(\mu^-(k))$ and $n = \min(\mu^-(j))$. Then $\mu(n) = j$ and $\mu(m) = k$ so, given $j > k$, we conclude $\mu(n) > \mu(m)$ and, thus, $n > m$ by monotonicity of μ. Take now any $x \in \mu^-(k)$. It is easy to see that $n > x$ since $n > m = \max(\mu^-(k))$. On the other hand, from $\mathbf{T}, \mu^-(j) \models \beta$ we conclude $\mathbf{T}, n \models \beta$. Now, as we know $\mathbf{M}, i \models \alpha$ for all $i \in [k..j)$, by induction hypothesis, $\mathbf{T}, h \models \alpha$ for all $h \in \mu^-(i)$. Finally, consider any point $y \in [x..n)$. As $n = \min(\mu^-(j))$ and $x \in \mu^-(k)$ and $n > x$ there must exist some $i \in [k..j)$ satisfying $\mu(y) = i$. But then, we had concluded $\mathbf{T}, h \models \alpha$ for $h \in \mu^-(i)$ and so, in particular $\mathbf{T}, y \models \alpha$. To sum up, we have some $n > x$ such that $\mathbf{T}, n \models \beta$ and $\mathbf{T}, y \models \alpha$ for all $y \in [x..n)$, that is, $\mathbf{T}, x \models \alpha \mathbf{U} \beta$, given any $x \in \mu^-(k)$.

- If $\varphi = \alpha \mathbf{R} \beta$, then $\mathbf{M}, k \models \alpha \mathbf{R} \beta$ and we want to prove $\mathbf{T}, \mu^-(k) \models \alpha \mathbf{R} \beta$ that is, for all $x \in \mu^-(k)$ and all $y \in [x..\lambda_t)$ such that $\mathbf{T}, y \not\models \beta$ we must find some $z \in [x..y)$ such that $\mathbf{T}, z \models \alpha$. Let us call $j = \mu(y)$. By monotonicity of μ, $y \geq x$ implies $j = \mu(y) \geq \mu(x) = k$. By the induction hypothesis, $\mathbf{T}, y \not\models \beta$ implies $\mathbf{M}, \mu(y) \not\models \beta$, that is, $\mathbf{M}, j \not\models \beta$ and $j \in [k..\lambda)$. Since $\mathbf{M}, k \models \alpha \mathbf{R} \beta$, this implies there is some $i \in [k..j]$ such that $\mathbf{M}, i \models \alpha$. By the induction hypothesis, $\mathbf{T}, \mu^-(i) \models \alpha$. By monotonicity of μ all points in $\mu^-(i)$ are greater or equal than $x \in \mu^-(k)$. Since $\mu^-(i)$ is not empty, we can take any $z \in \mu^-(i)$. But then, $x \leq z < y$ and $\mathbf{T}, z \models \alpha$ because $\mathbf{T}, \mu^-(i) \models \alpha$. Thus, $\mathbf{T}, x \models \alpha \mathbf{R} \beta$.

$\square$

Proof of Proposition 5. The left to right direction follows from Persistence (Theorem 1). For the right to left direction, we proceed by contradiction. Suppose $\mathbf{T}, \mu^-(k) \models \neg\varphi$ and $\mathbf{M}, k \not\models \neg\varphi$. The latter implies, as $\neg\varphi$ is $\varphi \to \bot$, that either (i) $\mathbf{M}, k \models \varphi$ and $\mathbf{M}, k \not\models \bot$; or (ii) $\mathbf{T}, i \models \varphi$ and $\mathbf{T}, i \not\models \bot$ for some $i \in \mu^-(k)$. In case (i), by Persistence $\mathbf{T}, \mu^-(k) \models \varphi$, hence $\mathbf{T}, j \not\models \neg\varphi$ for every $j \in \mu^-(k)$; as $\mu^-(k) \neq \emptyset$, this raises a contradiction. In case (ii), we similarly obtain $\mathbf{T}, i \not\models \neg\varphi$, which contradicts $\mathbf{T}, \mu^-(k) \models \neg\varphi$. $\square$

Proof of Proposition 6. We prove both directions by contraposition.

For the left to right direction, suppose $\not\models \varphi \to \psi$. This means there is some $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu \rangle$ and $k \in [0..|\mathbf{H}|)$ for which $\mathbf{M}, k \not\models \varphi \to \psi$. Then, we have two cases: (a) $\mathbf{M}, k \models \varphi$ and $\mathbf{M}, k \not\models \psi$; or (b) $\mathbf{T}, j \models \varphi$ and $\mathbf{T}, j \not\models \psi$ for some $j \in \mu^-(k)$. But in both cases, we found a trace and a point satisfying φ and falsifying ψ, something that implies $\varphi \not\models \psi$.

For the right to left direction, suppose $\varphi \not\models \psi$. This means there is a trace $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu \rangle$ and some $k \in [0..|\mathbf{H}|)$ for which $\mathbf{M}, k \models \varphi$ and $\mathbf{M}, k \not\models \psi$. But this implies $\mathbf{M}, k \not\models \varphi \to \psi$. $\square$

The following property can be easily observed by a simple analysis of cTHT satisfaction conditions:

Proposition 21 $\mathbf{M}, k \models \varphi$ *iff* $\mathbf{M}[k], 0 \models \varphi$.

Proof of Proposition 7. The left to right direction is trivial: if $\varphi \equiv \psi$ then $\mathbf{M}, k \models \varphi$ iff $\mathbf{M}, k \models \psi$ for any $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu \rangle$ and $k \in [0..|\mathbf{H}|)$ including the case $k = 0$.

For the right to left direction, we use contraposition. Suppose $\varphi \not\equiv \psi$ and, without loss of generality, assume that $\varphi \not\models \psi$. That is, $\mathbf{M}, k \models \varphi$ and $\mathbf{M}, k \not\models \psi$ for some $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu \rangle$ and $k \in [0..|\mathbf{H}|)$. By Proposition 25, we conclude $\mathbf{M}[k], 0 \models \varphi$ and $\mathbf{M}[k], 0 \not\models \psi$ and so, we found a trace $\mathbf{M}[k]$ that is a model of φ but not of ψ, i.e., the formulas do not have the same models. $\square$

Proof of Proposition 8. For the proof of $\Diamond$ we have the equivalent conditions:

$$
\begin{array}{ll}
 & \mathbf{M}, k \models \Diamond\varphi \\
\Leftrightarrow & \mathbf{M}, k \models \top \mathbf{U} \varphi \\
\Leftrightarrow & \text{for some } j \in [k..\lambda), \mathbf{M}, j \models \varphi \text{ and } \mathbf{M}, i \models \top \\
 & \text{for all } i \in [k..j) \\
\Leftrightarrow & \text{for some } j \in [k..\lambda), \mathbf{M}, j \models \varphi
\end{array}
$$

The proof of $\Box$ is analogous:

$$
\begin{array}{ll}
 & \mathbf{M}, k \models \Box\varphi \\
\Leftrightarrow & \mathbf{M}, k \models \bot \mathbf{R} \varphi \\
\Leftrightarrow & \text{for all } j \in [k..\lambda), \mathbf{M}, j \models \varphi \text{ or } \mathbf{M}, i \models \bot \\
 & \text{for some } i \in [k..j] \\
\Leftrightarrow & \text{for all } j \in [k..\lambda), \mathbf{M}, j \models \varphi
\end{array}
$$

The proof for $\mathbf{F}$ proceeds as follows. Let $\lambda_t = |\mathbf{T}|$.

$$
\begin{array}{lll}
 & \mathbf{M}, k \models \mathbf{F} & \\
\Leftrightarrow & \mathbf{M}, k \models \neg\circ\top & \\
\Leftrightarrow & \mathbf{T}, \mu^-(k) \models \neg\circ\top & \text{by Prop. 5} \\
\Leftrightarrow & \mathbf{T}, j \not\models \circ\top \text{for all } j \in \mu^-(k) & \\
\Leftrightarrow & \langle \mathbf{T}, \mathbf{T}, id \rangle, j \not\models \circ\top \text{for all } j \in \mu^-(k) & \\
\Leftrightarrow & \text{not } (\underbrace{|id^-(j)| = 1}_{\text{true}} \text{ and } j{+}1 < \lambda_t & \\
 & \quad \text{and } \underbrace{\mathbf{M}, j{+}1 \models \top}_{\text{true}}) & \\
 & \text{for all } j \in \mu^-(k) & \\
\Leftrightarrow & \text{not } (j{+}1 < \lambda_t) \text{ for all } j \in \mu^-(k) & \\
\Leftrightarrow & j{+}1 = \lambda_t \text{ for all } j \in \mu^-(k) &
\end{array}
$$

Now, note that the above condition states that j is the last position of $\mathbf{T}$ and this cannot be satisfied by more than one point $j \in \mu^-(k)$, so the latter is a singleton $\mu^-(k) = \{j\}$, namely:

$$
\begin{array}{lll}
\Leftrightarrow & j{+}1 = \lambda_t \text{ and } \mu^-(k) = \{j\} & \\
\Leftrightarrow & j{+}1 = \lambda_t \text{ and } \mu^-(k) = \{j\} & \text{By Lemma 3} \\
 & \text{and } k{+}1 = \lambda & \text{and } \mu(k) = j \\
\Leftrightarrow & \mu^-(k) = \{j\} \text{ and } k{+}1 = \lambda & \text{By Lemma 4} \\
\Leftrightarrow & |\mu^-(k)| = 1 \text{ and } k{+}1 = \lambda &
\end{array}
$$

Finally, the proof for $\widehat{\circ}$ follows from the equivalences

$$
\begin{array}{ll}
 & \mathbf{M}, k \models \widehat{\circ}\varphi \\
\Leftrightarrow & \mathbf{M}, k \models \circ\varphi \vee \mathsf{F} \\
\Leftrightarrow & (|\mu^-(k)| = 1 \text{ and } k{+}1 < \lambda \text{ and } \mathbf{M}, k{+}1 \models \varphi) \\
 & \text{or } (|\mu^-(k)| = 1 \text{ and } k{+}1 = \lambda) \\
\Leftrightarrow & |\mu^-(k)| = 1 \text{ and} \\
 & ((k{+}1 < \lambda \text{ and } \mathbf{M}, k{+}1 \models \varphi) \text{ or } k{+}1 = \lambda) \\
\Leftrightarrow & |\mu^-(k)| = 1 \text{ and} \\
 & \underbrace{(k{+}1 < \lambda \text{ or } k{+}1 = \lambda)}_{\text{true}} \\
 & \text{and } (\mathbf{M}, k{+}1 \models \varphi \text{ or } k{+}1 = \lambda) \\
\Leftrightarrow & |\mu^-(k)| = 1 \text{ and } (k{+}1 = \lambda \text{ or } \mathbf{M}, k{+}1 \models \varphi)
\end{array}
$$

□

Proof of Proposition 9. By definition of F, $\mathbf{M}, k \models \neg\mathsf{F}$ amounts to $\mathbf{M}, k \models \neg\neg\circ\top$. According to Proposition 5, this is equivalent to: $\mathbf{T}, j \models \neg\neg\circ\top$ in LTL, for all $j \in \mu^-(k)$. But this means $\mathbf{T}, j \models \circ\top$, for all $j \in \mu^-(k)$. The latter is trivially true for all elements $j \in \mu^-(k)$ except the maximum. Finally, if $j = \max(\mu^-(k))$, $\mathbf{T}, j \models \circ\top$ is equivalent to requiring that j is not the last position in $\mathbf{T}$, that is, $j + 1 < |\mathbf{T}|$. □ □

Lemma 6 *Let φ be a $\rightarrow$-free and $\circ$-free formula and let $\langle \mathbf{H}, \mathbf{T}, \mu\rangle$ and $\langle \mathbf{H}, \mathbf{T}', \mu'\rangle$ be two* HT*-traces. Then, $\langle \mathbf{H}, \mathbf{T}, \mu\rangle, k \models \varphi$ iff $\langle \mathbf{H}, \mathbf{T}', \mu'\rangle, k \models \varphi$.*

Proof. It is easy to see that, if the formula contains no implication or $\circ$ (or their derived operators), the satisfaction relation becomes independent of the $\mathbf{T}$ component or the contractor μ, so we may equivalently use any (well-formed) arbitrary $\mathbf{T}'$ and μ' without affecting satisfaction. □

By Lemma 6, if $\langle \mathbf{H}, \mathbf{T}, \mu\rangle, k \models \varphi$ then $\langle \mathbf{H}, \mathbf{H}, id\rangle \models \varphi$. Moreover, also By Lemma 6, the opposite holds too, and thus:

Corollary 4 *Let φ be a $\rightarrow$-free and $\circ$-free formula and let $\langle \mathbf{H}, \mathbf{T}, \mu\rangle$ be an* HT*-trace. Then, $\langle \mathbf{H}, \mathbf{T}, \mu\rangle, k \models \varphi$ iff $\mathbf{H}, k \models \varphi$.*

Proof of Proposition 10. The left to right direction follows directly from the fact cTHT $\subseteq$ LTL (Corollary 1). For the right to left direction, suppose $\varphi \leftrightarrow \psi$ is an LTL-tautology. Assume, for the sake of contradiction, that there exists $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu\rangle$ such that $\mathbf{M}, i \not\models \varphi \leftrightarrow \psi$. Then, note that $\mathbf{T}, \mu^-(i) \models \varphi \leftrightarrow \psi$ since we assumed this formula to be an LTL-tautology. As a result, the only possibilities are: either $\mathbf{M}, i \models \varphi$ and $\mathbf{M}, i \not\models \psi$; or $\mathbf{M}, i \models \psi$ and $\mathbf{M}, i \not\models \varphi$. Take the first case (the second is analogous): by Corollary 4, the condition amounts to $\mathbf{H}, i \models \varphi$ and $\mathbf{H}, i \not\models \psi$. But then, $\not\models \varphi \rightarrow \psi$ in LTL, and thus $\not\models \varphi \leftrightarrow \psi$ either, reaching a contradiction. □

Temporal Equilibrium Models

Proof of Theorem 2. Since THT models are also contracted ones with $\mu = id$, if $\mathbf{T}$ is an contracted temporal stable model, there is no smaller $\mathbf{H} \prec \mathbf{T}$ forming a THT model, and so, there is no integral smaller $\mathbf{H}$ either, $\mathbf{T}$ is also temporal stable model under the traditional or standard definition using integral pairs of traces. □

$\circ$-free formulas

Proof of Lemma 1. Let $\mathbf{T}$ (resp. $\mathbf{H}$) be a stuttering of $\mathbf{T}'$ (resp. $\mathbf{H}'$), then the 1. point is a well-known result of LTL that can be proved on the structural complexity of φ.

Point 2. can be reduce to 1. using the *star-transformation* for THT formulas (Aguado et al. 2008) here recalled:

- $(\bot)^* = \bot$ and $(p)^* = p'$, where p' is a fresh atom;
- $(\varphi_1 \cdot \varphi_2)^* = (\varphi_1)^* \cdot (\varphi_2)^*$ for $\cdot \in \{\vee, \wedge, \mathsf{U}, \mathsf{R}\}$;
- $(\cdot\, \varphi)^* = \cdot\, (\varphi)^*$ for $\cdot \in \{\Box, \Diamond\}$, and
- $(\varphi_1 \rightarrow \varphi_2)^* = (\varphi_1^* \rightarrow \varphi_2^*) \wedge (\varphi_1 \rightarrow \varphi_2)$.

From Theorem 1 in (Aguado et al. 2008), we know that $\langle \mathbf{H}, \mathbf{T}\rangle \models \varphi$ iff $\mathbf{T}' \models_{\mathrm{LTL}} \varphi^* \wedge \Box(\bigwedge_{p \in \mathcal{A}} p' \rightarrow p)$, where $p \in \mathbf{T}'_i$ iff $p \in \mathbf{T}_i$ and $p' \in \mathbf{T}'_i$ iff $p \in \mathbf{H}_i$ for each $p \in \mathcal{A}$ and $i \geq 0$.

□

Proof of Proposition 11. The proof is by structural induction. Let $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu\rangle$.

The cases 1-4 in Definition 7 are immediately verified; in particular case 2 (φ is an atom p) is by the construction of $\mathbf{H}$ from $\mathbf{T}'$.

In case 5, φ is of the form $\varphi_1 \rightarrow \varphi_2$, and we have

- $\mathbf{M}, k \models \varphi$ iff $\begin{cases} 1a: & \mathbf{M}, k \not\models \varphi_1 \text{ or } \mathbf{M}, k \models \varphi_2 \\ 1b: & \langle \mathbf{T}, \mathbf{T}, id\rangle, j \not\models \varphi_1 \text{ or} \\ & \langle \mathbf{T}, \mathbf{T}, id\rangle, j \models \varphi_2 \text{ for all } j \in \mu^{-1}(k) \end{cases}$
- $\langle \mathbf{H}, \mathbf{T}\rangle, i \models \varphi$ iff $\begin{cases} 2a: \langle \mathbf{H}, \mathbf{T}\rangle, i \not\models \varphi_1 \text{ or } \langle \mathbf{H}, \mathbf{T}\rangle, i \models \varphi_2 \\ 2b: \langle \mathbf{T}, \mathbf{T}\rangle, i \not\models \varphi_1 \text{ or } \langle \mathbf{T}, \mathbf{T}\rangle, i \models \varphi_2 \end{cases}$

By the induction hypothesis, conditions $1a$ and $2a$ are equivalent. Furthermore, also conditions $1b$ and $2b$ are equivalent, as by Proposition 4 for any j and formula ψ, $\langle \mathbf{T}, \mathbf{T}\rangle, j \models \psi$ iff $\langle \mathbf{T}, \mathbf{T}, id\rangle, j \models \psi$, and we can apply the induction hypothesis for $\psi \in \{\varphi_1, \varphi_2\}$ and obtain that $\langle \mathbf{T}, \mathbf{T}\rangle, i \models \psi$ iff $\langle \mathbf{T}, \mathbf{T}\rangle, j \models \psi$, for all $j \in \mu^-(k)$.

Case 6 does not apply, as φ is $\circ$-free.

In case 7, i.e., $\varphi = \varphi_1 \mathbin{\mathsf{U}} \varphi_2$, $\mathbf{M}, k \models \varphi$ iff for some $j \in [k..\lambda)$, we have $\mathbf{M}, j \models \varphi_2$ and $\mathbf{M}, j' \models \varphi_1$ for all $j' \in [k..j)$.

By the induction hypothesis, we have for each $j' \in [k..j)$ that (a) $\mathbf{M}, j' \models \varphi_1$ iff $\langle \mathbf{H}, \mathbf{T}\rangle, i \models \varphi_1$ for some (equivalently, every) $i \in \mu^-(j')$, and (b) $\mathbf{M}, j \models \varphi_2$ iff $\langle \mathbf{H}, \mathbf{T}\rangle, i \models \varphi_2$ for some (equivalently, every) $i \in \mu^-(j)$.

Suppose $\mathbf{M}, k \models \varphi$. If $j = k$, we must have $\langle \mathbf{H}, \mathbf{T}\rangle, i \models \varphi_2$ and thus $\langle \mathbf{H}, \mathbf{T}\rangle, i \models \varphi$; if $j > k$, then $\langle \mathbf{H}, \mathbf{T}\rangle, i' \models \varphi_1$ for all $i' \in [i, i'')$ and $\langle \mathbf{H}, \mathbf{T}\rangle, i'' \models \varphi_2$, where $i'' = \min(\mu^-(j))$, thus again $\langle \mathbf{H}, \mathbf{T}\rangle, i \models \varphi$. Conversely, suppose $\langle \mathbf{H}, \mathbf{T}\rangle, i \models \varphi$. If $\langle \mathbf{H}, \mathbf{T}\rangle, i \models \varphi_2$, then we can set $j = k$ and obtain by (b) that $\mathbf{M}, k \models \varphi$. Otherwise, we set $j > k$, where $j = \mu(i')$ and $i' \geq i$ is the least number such that $\langle \mathbf{H}, \mathbf{T}\rangle, i' \models \varphi_2$; then $\mathbf{M}, j' \models \varphi_1$ holds by (a) for each $j' \in [k..j)$ and $\mathbf{M}, j \models \varphi_2$ holds by (b), thus $\mathbf{M}, k \models \varphi$.

Case 8, i.e., $\varphi = \varphi_1 \mathbin{\mathsf{R}} \varphi_2$, is similar to the previous case, where we may consider the equivalent definition that $\mathbf{M}, k \models \varphi$ iff either $\mathbf{M}, j \models \varphi_2$ for all $j \in [k..\lambda)$ or for some $j \in [k..\lambda)$, we have $\mathbf{M}, j \models \varphi_1$ and $\mathbf{M}, j' \models \varphi_2$ for all $j' \in [k..j]$.

$\square$

Proof of Proposition 12. Let $\mathbf{T}$ be an c-stable model of Γ. Towards a contradiction, suppose $\mathbf{T}$ is a proper stuttering of some trace $\mathbf{T}'$, i.e., $\mathbf{T} \neq \mathbf{T}'$ and $\mathbf{T} \downarrow^{\mu} \mathbf{T}'$ for some μ such that $T_i = T'_{\mu(i)}$ for each $i \in [0..\lambda)$. By Proposition 11, we obtain for $\mathbf{H} = \mathbf{T}$ that $\langle \mathbf{T}', \mathbf{T}, \mu \rangle, 0 \models \varphi$ iff $\langle \mathbf{T}, \mathbf{T} \rangle, 0 \models \varphi$ for every formula $\varphi \in \Gamma$, thus $\langle \mathbf{T}', \mathbf{T}, \mu \rangle \models \Gamma$ iff $\langle \mathbf{T}, \mathbf{T} \rangle \models \Gamma$. As $\mathbf{T}$ is c-stable, by Theorem 2 it is an stable model of Γ, and thus $\langle \mathbf{T}, \mathbf{T} \rangle \models \Gamma$ holds. Consequently $\langle \mathbf{T}', \mathbf{T}, \mu \rangle \models \Gamma$, which means that $\mathbf{T}$ is not c-stable, which is the desired contradiction. $\square$

Proof of Proposition 13. We show this result by structural induction, where we take into account that for positive formulas, the stronger condition of Proposition 11 holds, i.e.,

(*) for every $k \in [0, \lambda')$ and $i \in \mu^{-}(k)$, $\langle \mathbf{T}', \mathbf{T}, \mu \rangle, k \models \varphi$ iff $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi$.

This can be easily shown along the lines of the proof of Proposition 11; informally, for positive φ the $\mathbf{T}$ trace is irrelevant.

The cases 1 and 2 in Definition 7 are thus covered, as there φ is positive.

In case 3, $\varphi = \varphi_1 \wedge \varphi_2$ and $\mathbf{M} \models \varphi$ implies $\mathbf{M} \models \varphi_1$ and $\mathbf{M} \models \varphi_2$. By the induction hypothesis, $\mathbf{M} \models \varphi_j$ implies $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi_j$ for every $i \in \mu^{-}(k)$ and $j \in \{1, 2\}$; consequently, $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi_1 \wedge \varphi_2$, i.e., $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi$, for every $i \in \mu^{-}(k)$.

In case 4, $\varphi = \varphi_1 \vee \varphi_2$ and $\mathbf{M} \models \varphi$ implies either $\mathbf{M} \models \varphi_1$ or $\mathbf{M} \models \varphi_2$. By the induction hypothesis, $\mathbf{M} \models \varphi_j$ implies $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi_j$ for every $i \in \mu^{-}(k)$ and $j \in \{1, 2\}$. By weakening φ_j to $\varphi = \varphi_1 \vee \varphi_2$, it follows that $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi$ for every $i \in \mu^{-}(k)$.

In case 5, $\varphi = \varphi_1 \rightarrow \varphi_2$, and as φ is from THT^1, φ_1 and φ_2 are positive. We recall from the proof of Proposition 11 the conditions of satisfactions:

- $\mathbf{M}, k \models \varphi$ iff $\begin{cases} 1a: & \mathbf{M}, k \not\models \varphi_1 \text{ or } \mathbf{M}, k \models \varphi_2 \\ 1b: & \langle \mathbf{T}, \mathbf{T}, id \rangle, j \not\models \varphi_1 \text{ or} \\ & \langle \mathbf{T}, \mathbf{T}, id \rangle, j \models \varphi_2 \text{ for all } j \in \mu^{-1}(k) \end{cases}$
- $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi$ iff $\begin{cases} 2a: \langle \mathbf{H}, \mathbf{T} \rangle, i \not\models \varphi_1 \text{ or } \langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi_2 \\ 2b: \langle \mathbf{T}, \mathbf{T} \rangle, i \not\models \varphi_1 \text{ or } \langle \mathbf{T}, \mathbf{T} \rangle, i \models \varphi_2 \end{cases}$

By the stronger property (*) for φ_1 and φ_2, the conditions $1a$ and $2a$ are equivalent. Furthermore, clearly condition $1b$ implies condition $2b$; hence $\mathbf{M} \models \varphi$ implies $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi$ for all $i \in \mu^{-}(k)$.

Case 6 does not apply, as φ is $\circ$-free.

In case 7, i.e., $\varphi = \varphi_1 \,\mathbf{U}\, \varphi_2$, $\mathbf{M}, k \models \varphi$ implies that for some $j \in [k..\lambda)$, we have $\mathbf{M}, j \models \varphi_2$ and $\mathbf{M}, j' \models \varphi_1$ for all $j' \in [k..j)$. By the induction hypothesis, we have for each $j' \in [k..j)$ that $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi_1$ for every $i \in \mu^{-}(j')$ and $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi_2$ for every $i \in \mu^{-}(j)$. If $j = k$, we have $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi_2$ and thus $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi$, for every $i \in \mu^{-}(k)$. If $j > k$, then $\langle \mathbf{H}, \mathbf{T} \rangle, i' \models \varphi_1$ for all $i' \in [i, i'')$ and $\langle \mathbf{H}, \mathbf{T} \rangle, i'' \models \varphi_2$, where $i'' = \min(\mu^{-}(j))$, thus again $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi$ for every $i \in \mu^{-}(k)$.

Case 8, i.e., $\varphi = \varphi_1 \,\mathbf{R}\, \varphi_2$, is again similar to the previous case, using that $\mathbf{M}, k \models \varphi$ iff either $\mathbf{M}, j \models \varphi_2$ for all $j \in [k..\lambda)$ or for some $j \in [k..\lambda)$, we have $\mathbf{M}, j \models \varphi_1$ and $\mathbf{M}, j' \models \varphi_2$ for all $j' \in [k..j)$. $\square$

Proof of Proposition 14. Assume $\mathbf{T}$ is an stable model of Γ and not a stuttering of any sequence $\mathbf{T}' \neq \mathbf{T}$. Towards a contradiction, suppose $\mathbf{T}$ is not c-stable. Then some $\mathbf{T}' \neq \mathbf{T}$ and μ exist such that $\mathbf{T} \downarrow^{\mu} \mathbf{T}'$ and $\langle \mathbf{T}', \mathbf{T}, \mu \rangle \models \Gamma$. Let $\mathbf{H}$ be the stuttering of $\mathbf{T}'$ induced by μ, i.e., such that $\mathbf{H} \downarrow^{\mu} \mathbf{T}'$. Then by Proposition 13, $\langle \mathbf{H}, \mathbf{T} \rangle \models \Gamma$. As $\mathbf{T} \downarrow^{id} \mathbf{H}$, we must have $\mathbf{H} = \mathbf{T}$; otherwise, $\mathbf{T}$ would not be stable. This, however, implies that $T'_j = T_i$ for each $j \in [0, \lambda')$ and $i \in \mu^{-}(j)$. As $\mathbf{T} \neq \mathbf{T}'$. this means that $\mathbf{T}$ is a stuttering of $\mathbf{T}'$, which is in contradiction to the assertion. $\square$

Proof of Theorem 3. It remains to argue about items (ii) and (iii) of the statement. As for (ii), suppose $\mathbf{T}$ is an stable model of Γ; then we have $\langle \mathbf{T}, \mathbf{T} \rangle \models \Gamma$. Let us contract $\mathbf{T}$ to the sequence $\mathbf{T}'$ by mapping each maximal segment of identical elements to the first element, i.e., $\mu(0) = 0$ and $T'_0 = T_0$, and for each $i \in [1..\lambda)$,

$$\mu(i+1) = \begin{cases} \mu(i) & \text{if } T_{i+1} = T_i, \text{ and} \\ \mu(i+1) = \mu(i) + 1 & \text{otherwise, with} \\ & T'_{\mu(i)+1} = T_{i+1}. \end{cases}$$

Then we obtain from Proposition 11 that $\langle \mathbf{T}', \mathbf{T}, \mu \rangle \models \Gamma$ ($\mathbf{T}$ itself is the stuttering $\mathbf{H}$ of $\mathbf{T}'$ there, i.e. $\mathbf{H} = \mathbf{T}$).

Let us assume towards a contradiction that $\mathbf{T}'$ is not stable, namely that there exists $\mathbf{H}'$ such that $\langle \mathbf{H}', \mathbf{T}' \rangle \models \Gamma$, and $\mathbf{H}' \subset \mathbf{T}'$. By Lemma 1, we can stretch $\mathbf{T}'$ into $\mathbf{T}$ and similarly $\mathbf{H}'$ into $\mathbf{H}$, so that we obtain $\langle \mathbf{H}, \mathbf{T} \rangle \models \Gamma$; this contradicts the hypothesis that $\mathbf{T}$ was stable. Hence, (ii) holds. Item (iii) then holds since by construction $\mathbf{T}'$ is stutter-free. $\square$

GP-formulas

For our purposes, the following lemmas are useful.

Lemma 7 *Suppose φ is a positive formula. Then for every $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ and every $k \in [0..\lambda)$, it holds that $\mathbf{M}, k \models \varphi$ iff $\mathbf{H}, k \models \varphi$ iff $\mathbf{H}, k \models \varphi$ and $\mathbf{T}, k \models \varphi$.*

Proof. (Sketch) The proof is by structural induction on φ, using the fact that $\mathbf{T}$ plays no role for the evaluation of a positive formula φ, and using Persistence. $\square$

Lemma 8 *For every $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ and $k \in [0..\lambda)$, (1) if φ is from THT^1 then $\mathbf{M}, k \models \varphi$ implies $\mathbf{H}, k \models \varphi$, and (2) if φ is from GP, then $\mathbf{M}, k \models \varphi$ iff $\mathbf{H}, k \models \varphi$ and $\mathbf{T}, k \models \varphi$.*

Proof. The proof is by structural induction. We first show (1).

The cases 1 ($\varphi = \bot, \top$), 2 ($\varphi = p$) and 3 ($\varphi = \varphi_1 \wedge \varphi_2$, using the induction hypothesis) in Definition 7 are immediate. In case 4, if $\mathbf{M}, k \models \varphi_1 \vee \varphi_2$ then w.l.o.g. $\varphi_1 \neq \bot$ (else the result follows immediately by the induction hypothesis for φ_2) and $\mathbf{M}, k \models \varphi_1$. $\mathbf{H}, k \models \varphi_1$ holds by the induction hypothesis. Thus $\mathbf{H}, k \models \varphi_1 \vee \varphi_2$.

In case 5, φ is of the form $\varphi_1 \to \varphi_2$, where φ_1 and φ_2 are positive. We show the claim by contraposition. Suppose that $\mathbf{H}, k \not\models \varphi$, i.e., $\mathbf{H}, k \models \varphi_1$ and $\mathbf{H}, k \not\models \varphi_2$. As φ_1 and φ_2 are positive, by Lemma 7 $\mathbf{M}, k \models \varphi_1$ and $\mathbf{M}, k \not\models \varphi_2$; hence $\mathbf{M}, k \not\models \varphi$.

In case 6, $\mathbf{M}, k \models \circ\varphi$ iff $\mathbf{M}, k{+}1 \models \varphi$ and $k{+}1 < \lambda$; thus by the induction hypothesis, $\mathbf{M}, k \models \circ\varphi$ implies $\mathbf{H}, k{+}1 \models \varphi$ which is equivalent to $\mathbf{H}, k \models \circ\varphi$.

In case 7, $\varphi = \varphi_1 \ \mathbf{U}\ \varphi_2$ and $\mathbf{M}, k \models \varphi$ iff for some $j \in [k..\lambda)$, we have $\mathbf{M}, j \models \varphi_2$ and $\mathbf{M}, j' \models \varphi_1$ for all $j' \in [k..j)$. By the induction hypothesis, it follows that $\mathbf{H}, j \models \varphi_2$ and $\mathbf{H}, j' \models \varphi_2$ for each $j' \in [k..j)$; hence, $\mathbf{H}, k \models \varphi$.

Case 8, i.e., $\varphi = \varphi_1 \ \mathbf{R}\ \varphi_2$, is similar to the previous case, where we may consider the equivalent definition that $\mathbf{M}, k \models \varphi$ iff either $\mathbf{M}, j \models \varphi_2$ for all $j \in [k..\lambda)$ or for some $j \in [k..\lambda)$, we have $\mathbf{M}, j \models \varphi_1$ and $\mathbf{M}, j' \models \varphi_2$ for all $j' \in [k..j)$.

This concludes the proof of (1). For (2), it remains to prove the if-direction for GP-formulas, i.e., that for every $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ and k it holds that $\mathbf{H}, k \models \varphi$ and $\mathbf{T}, k \models \varphi$ implies $\mathbf{M}, k \models \varphi$. We again proceed by structural induction.

As above the cases 1 ($\varphi = \bot, \top$), 2 ($\varphi = p$) and 3 ($\varphi = \varphi_1 \wedge \varphi_2$, using the induction hypothesis) are immediate.

In case 4, consider $\mathbf{H}, k \models \varphi_1 \vee \varphi_2$. Then w.l.o.g. $\mathbf{H}, k \models \varphi_1$; as φ_1 must be positive, by Lemma 7 $\mathbf{M}, k \models \varphi_1$, thus by weakening also $\mathbf{M}, k \models \varphi_1 \vee \varphi_2$.

In case 5, we argue by contraposition. If $\mathbf{M}, k \not\models \varphi$, then either (a) $\mathbf{M}, k \models \varphi_1$ and $\mathbf{M}, k \not\models \varphi_2$ or (b) $\mathbf{T}, k \models \varphi_1$ and $\mathbf{T}, k \not\models \varphi_2$, i.e., $\mathbf{T}, k \not\models \varphi$. It remains to consider (a), where we can assume $\mathbf{T}, k \models \varphi$. As φ_1 is positive, it follows by Lemma 7 from $\mathbf{M}, k \models \varphi_1$ that $\mathbf{H}, k \models \varphi_1$ and $\mathbf{T}, k \models \varphi_1$, which implies $\mathbf{T}, k \models \varphi_2$ as $\mathbf{T}, k \models \varphi$. Since $\mathbf{M}, k \not\models \varphi_2$ and φ_2 is positive, it follows by Lemma 7 that $\mathbf{H}, k \not\models \varphi_2$. Hence, $\mathbf{H}, k \not\models \varphi$.

In case 6, $\mathbf{M}, k \models \circ\varphi$ iff $\mathbf{M}, k{+}1 \models \varphi$ and $k{+}1 < \lambda$. If $\mathbf{H}, k \models \circ\varphi$ and $\mathbf{T}, k \models \circ\varphi$, then $\mathbf{H}, k{+}1 \models \varphi$ and $\mathbf{T}, k{+}1 \models \varphi$, and by the induction hypothesis hypothesis $\mathbf{M}, k{+}1 \models \varphi$, which implies hypothesis $\mathbf{M}, k \models \circ\varphi$.

In case 7, suppose $\mathbf{H}, k \models \varphi$, and $\mathbf{T}, k \models \varphi$. Then some $j \in [k..\lambda)$ exists such that $\mathbf{H}, j \models \varphi_2$ and $\mathbf{H}, j' \models \varphi_1$ for all $j' \in [k..j)$. If $\varphi_1 = \bot$, then $j = k$ and φ is equivalent to φ_2, and the result follows from the induction hypothesis. Otherwise, φ_1 and φ_2 are positive, and by Lemma 7 $\mathbf{T}, j \models \varphi_2$ and $\mathbf{T}, j' \models \varphi_1$; hence by the induction hypothesis $\mathbf{M}, j \models \varphi_2$ and $\mathbf{M}, j' \models \varphi_1$ for all $j' \in [k..j)$, thus $\mathbf{M}, k \models \varphi$.

The case 8, i.e., $\varphi = \varphi_1 \ \mathbf{R}\ \varphi_2$, is similar to case 7, using that $\mathbf{M}, k \models \varphi$ iff either $\mathbf{M}, j \models \varphi_2$ for all $j \in [k..\lambda)$ or for some $j \in [k..\lambda)$, we have $\mathbf{M}, j \models \varphi_1$ and $\mathbf{M}, j' \models \varphi_2$ for all $j' \in [k..j)$.

□

Note that item (2) of Lemma 8 would not hold for $\neg p \vee p$ nor for $\neg p \to p$ or $\Diamond(p \to q)$; the latter is witnessed by $\mathbf{T} = \{p\}\cdot\{p, q\}$ and $\mathbf{H} = \emptyset\cdot\{p\}$, as both $\mathbf{T}$ and $\mathbf{H}$ are models of $\Diamond(p \to q)$ while $\langle \mathbf{H}, \mathbf{T} \rangle$ is not a model. Item (1) was reported for infinite HT-traces in (Bozzelli and Pearce 2015).

Lemma 8 can be generalized for $\circ$-free formulas to contracted HT-traces.

Proposition 22 *For every* HT-*trace* $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu \rangle$*,* $\circ$-*free formula* φ*, and* $k \in [0..\lambda')$*, (1) if* φ *is from* THT^1*, then* $\mathbf{M}, k \models \varphi$ *implies* $\mathbf{T}', k \models \varphi$*, and (2) if* φ *is from* GP*, then* $\mathbf{M}, k \models \varphi$ *iff* $\mathbf{T}', k \models \varphi$ *and* $\mathbf{T}, \mu^-(k) \models \varphi$.

Proof. Item (1) follows from Proposition 13 and Lemma 8, as for the stuttering $\mathbf{H}$ of $\mathbf{T}'$ such that $\mathbf{H} \downarrow^\mu \mathbf{T}'$ we have $\langle \mathbf{H}, \mathbf{T} \rangle, i \models \varphi$ for every $i \in \mu^-(k)$ and by Lemma 8 that $\mathbf{H}, i \models \varphi$; by Lemma 1, it follows that $\mathbf{T}', k \models \varphi$.

As for (2) the only-if directions follows from (1) and Persistence (Theorem 1). The if-direction is shown by structural induction, where we use for positive formulas φ the weaker condition $\mathbf{T}', k \models \varphi$.

The cases 1 ($\varphi = \bot, \top$), 2 ($\varphi = p$) and 3 ($\varphi = \varphi_1 \wedge \varphi_2$, using the induction hypothesis) in Definition 7 are immediate.

In case 4, consider $\mathbf{T}', k \models \varphi_1 \vee \varphi_2$. Then w.l.o.g. $\mathbf{T}', k \models \varphi_1$; as φ_1 must be positive, by the induction hypothesis $\mathbf{M}, k \models \varphi_1$, thus by weakening also $\mathbf{M}, k \models \varphi_1 \vee \varphi_2$.

In case 5, we argue by contraposition. If $\mathbf{M}, k \not\models \varphi$, then either (a) $\mathbf{M}, k \models \varphi_1$ and $\mathbf{M}, k \not\models \varphi_2$ or (b) $\mathbf{T}, \mu^-(k) \models \varphi_1$ and $\mathbf{T}, \mu^-(k) \not\models \varphi_2$, i.e., $\mathbf{T}, \mu^-(k) \not\models \varphi$. It remains to consider (a), where we can assume $\mathbf{T}, \mu^-(k) \models \varphi$. As φ_1 is positive, it follows by Lemma 7 from $\mathbf{M}, k \models \varphi_1$ that $\mathbf{T}', k \models \varphi_1$ and $\mathbf{T}, \mu^-(k) \models \varphi_1$, which implies $\mathbf{T}, \mu^-(k) \models \varphi_2$ as $\mathbf{T}, \mu^-(k) \models \varphi$. Since $\mathbf{M}, k \not\models \varphi_2$ and φ_2 is positive, it follows by Lemma 7 that $\mathbf{T}', k \not\models \varphi_2$. Hence, $\mathbf{T}', k \not\models \varphi$.

In case 7, suppose $\mathbf{T}', k \models \varphi$, and $\mathbf{T}, \mu^-(k) \models \varphi$. Then some $j \in [k..\lambda')$ exists such that $\mathbf{T}', j \models \varphi_2$ and $\mathbf{T}', j' \models \varphi_1$ for all $j' \in [k..j)$. If $\varphi_1 = \bot$, then $j = k$ and φ is equivalent to φ_2, and the result follows from the induction hypothesis. Otherwise, φ_1 and φ_2 are positive, and by the induction hypothesis $\mathbf{M}, j \models \varphi_2$ and $\mathbf{M}, j' \models \varphi_1$ for all $j' \in [k..j)$, thus $\mathbf{M}, k \models \varphi$.

In case 8, $\varphi = \varphi_1 \mathbf{R} \varphi_2$. We use that $\mathbf{M}, k \models \varphi$ iff either (a) $\mathbf{M}, j \models \varphi_2$ for all $j \in [k..\lambda')$ or (b) for some $j \in [k..\lambda')$, we have $\mathbf{M}, j \models \varphi_1$ and $\mathbf{M}, j' \models \varphi_2$ for all $j' \in [k..j]$. Assume $\mathbf{T}', k \models \varphi$ and $\mathbf{T}, \mu^-(k) \models \varphi$.

If $\varphi_1 = \bot$, then $\mathbf{T}', j \models \varphi_2$ and $\mathbf{T}, \mu^-(j) \models \varphi_2$ for all $j \in [k..\lambda')$. Thus by the induction hypothesis, $\mathbf{M}, j \models \varphi_2$ for all $j \in [k..\lambda')$, which by (a) implies $\mathbf{M}, k \models \varphi$. Otherwise, $\varphi_1 \neq \bot$. As $\mathbf{T}', k \models \varphi$, either (a′) $\mathbf{T}', j \models \varphi_2$ for all $j \in [k..\lambda')$ or (b′) for some $j \in [k..\lambda')$, we have $\mathbf{T}', j \models \varphi_1$ and $\mathbf{T}', j' \models \varphi_2$ for all $j' \in [k..j]$. Since both φ_1 and φ_2 are positive, using the induction hypothesis (a′) implies (a) and (b′) implies (b); hence, $\mathbf{M}, k \models \varphi$.

□

The relation between $\subseteq$-minimal models and stable models of THT^1 resp. GP theories is then as follows.

Corollary 5 *For any theory* Γ *of* THT^1 *(resp.,* GP*) formulas,* $\mathbf{T}$ *is an stable model of* Γ *if (resp., if and only if)* $\mathbf{T}$ *is a* $\subseteq$*-minimal model of* Γ.

Proof of Theorem 4. The equivalence of (1) and (2) follows from Theorem 3 and Corollary 5. That (3) implies (1) is shown using Proposition 22. Let $\mathbf{T}$ be a model of Γ and assume that no model $\mathbf{T}'$ of Γ such that $\mathbf{T}' \prec \mathbf{T}$ exists. Towards a contradiction, assume that $\mathbf{T}$ is not c-stable. The latter implies that some $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu \rangle$ exists such that $\mathbf{T}' \neq \mathbf{T}$ and $\mathbf{M} \models \Gamma$, which by Proposition 22 is equivalent to $\mathbf{T}', 0 \models \Gamma$ and $\mathbf{T}, \mu^{-}(0) \models \Gamma$. As $\mathbf{T}' \neq \mathbf{T}$ implies $\mathbf{T}' \prec \mathbf{T}$, this is a contradiction. Finally, we show that (2) implies (3). Assume that $\mathbf{T}$ is a minimal model of Γ that is stutter-free. Towards a contradiction, assume that $\mathbf{T}' \models \Gamma$ for some $\mathbf{T}' \prec \mathbf{T}$, i.e., $\mathbf{T} \downarrow^{\mu} \mathbf{T}'$ for some μ. Let $\mathbf{H}$ be the stuttering of $\mathbf{T}'$ such that $\mathbf{H} \downarrow^{\mu} \mathbf{T}'$. Then $\mathbf{H} \models \Gamma$ (cf. Pumping Lemma 1), and by Lemma 8, it follows that $\langle \mathbf{H}, \mathbf{T} \rangle \models \Gamma$. By minimality of $\mathbf{T}$, it follows that $\mathbf{H} = \mathbf{T}$; thus the only way in which we may still have $\mathbf{T}' \prec \mathbf{H} = \mathbf{T}$ is if $\mathbf{H} = \mathbf{T}$ is not stutter-free, which again is a contradiction. □

Proof of Proposition 15. As for (1), by Corollary 5, $\mathbf{T}$ is an stable model of Γ if $\mathbf{T}$ is a $\subseteq$-minimal model of Γ, which then by Theorem 3 is c-stable if $\mathbf{T}$ is stutter-free. As for (2), suppose $\mathbf{T} \models \Gamma$ and no model $\mathbf{T}' \prec \mathbf{T}$ of Γ exists. If $\mathbf{T}$ would not be c-stable, then $\langle \mathbf{T}', \mathbf{T}, \mu \rangle \models \Gamma$ for some $\mathbf{T}'$ and μ such that $\mathbf{T}' \neq \mathbf{T}$, thus $\mathbf{T}' \prec \mathbf{T}$. By Proposition 22, $\mathbf{T}' \models \Gamma$ which is a contradiction. □

Taming summarization

Proof of Proposition 17. It is an immediate result since given a $\mathbf{T}$ LTL model of Γ, the admissible defeaters $\langle \mathbf{H}, \mathbf{T}, \mu \rangle$ of Defn. 12 are a subset of the admissible defeaters of Defn. 9. □

Proof of Theorem 6. Given a $\circ$-free theory Γ, let we know that T-stable models of Γ are stutter-free. This can be shown as in Proposition 12 by replacing c-stability with T-stability in the proof.

We need to show that stutter-free stable traces are T-stable. If $\mathbf{T}$ is a stutter-free stable trace of Γ but not T-stable, then by Proposition 11 there exists a T-stuttering model $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu \rangle$ of Γ such that for all $i \neq j \in \mu^{-}(k)$ $T_i = T_j$ and $\mathbf{T} \neq \mathbf{T}'$. However, one of our hypothesis was the stutter-freeness of $\mathbf{T}$, therefore the only candidates to be T-stuttering model can be contracted via id. However, if we have $\langle \mathbf{T}', \mathbf{T}, id \rangle$ and $\mathbf{T}' \neq \mathbf{T}$ there must be an index $i \in [0, \lambda)$ such that $T'_i \neq T_i$ which would contradict the hypothesis of $\mathbf{T}$ being a stable model of Γ.

The second part of Theorem 6 can be proved following the same argument of the second part of Theorem 3. □

Proof of Theorem 5. Given a theory Γ, we want to show that the c-stable models of $\Gamma \cup \Box(\mathbf{WEM})$ coincide with the T-stable models of Γ.

We first notice that $\Box(\mathbf{WEM})$ is a tautology for T-stable models. In fact, given a T-stutter interpretation $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu \rangle$, $\Box(\neg p \vee \neg\neg p)$ requires that for each $i, j \in \mu^{-}(k)$ either $p \in T_i \cup T_j$ or that $p \notin T_i \cap T_j$.

By Proposition 17 we know that if $\mathbf{T}$ is a c-stable of $\Gamma \cup \Box(\mathbf{WEM})$ then $\mathbf{T}$ is also a T-stable model of $\Gamma \cup \Box(\mathbf{WEM})$. However, thanks to the previous observation $\Box(\mathbf{WEM})$ is a tautology for T-stutter models, therefore we get that $\mathbf{T}$ is a T-stable model of Γ.

For the other direction, let us assume that $\mathbf{T}$ is a T-stable model of Γ, then it is also a T-stable model of $\Gamma \cup \Box(\mathbf{WEM})$. Again, thanks to the previous observation, $\Box(\mathbf{WEM})$ forces cTHT interpretations to be T-stutter interpretations, therefore candidates in Defn. 9 can be restricted to T-stutter interpretations, and the definition collapses to Defn. 12. □

Proof of Proposition 16. It is immediate to see that a T-stuttering $\circ$-free tautology is also a THT tautology.

Let us consider a $\circ$-free THT tautology φ and let us consider for the sake of contradiction that there exists a well formed T-stutter interpretation $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu \rangle$ such that $\mathbf{M} \not\models \varphi$. Then by Proposition 11 we know that if $\mathbf{M}$ is a T-stuttering, then we have $\mathbf{M}, \mu(k) \models \varphi$ iff $\langle \mathbf{H}, \mathbf{T} \rangle \models \varphi$ with $\mathbf{H}$ stuttering of $\mathbf{T}'$ via μ. Which means that φ is not a THT tautology. Therefore, we reached the desired conclusion. □

Computational Complexity

Lemma 9 TEL *satisfiability for* $\mathrm{GP}^1_2(\Box, \Diamond)$ *formulas for infinite traces is EXPSPACE-hard.*

Proof. In order to prove the theorem, we exploit and adapt the technique used in (Bozzelli and Pearce 2016) showing that $\mathrm{THT}^1_2(\Diamond, \Box)$ is EXPSPACE-complete.

We cannot directly apply their reduction since they use a disjunction of implications, which is not allowed as a GP formula. However, we can exploit a similar schema.

We use a a polynomial reduction from an EXPSPACE-complete version of the *tiling* problem (van Emde Boas 2019).

Let $\mathcal{I}$ be an instance such that $\mathcal{I} = \{C, \Delta, n, d_{init}, d_{final}\}$, where C is a finite set of colors, $\Delta \subseteq C^4$ is the set of *domino types*, $n > 0$ is a natural number written in unary, and $d_{init}, d_{final} \in \Delta$ are domino types, and $|\Delta| = m$. We denote by $left : \Delta \to C$ (respectively, $right$, up, $down$) the color of the left edge of the domino type. A tiling is a mapping $f : [0, k] \times [0, 2^n - 1] \to \Delta$ for some $k \geq 0$ such that:

- for all $(i, j), (i, j+1) \in [0, k] \times [0, 2^n - 1]$ with $j < 2^n - 1$, $right(f(i, j)) = left(f(i, j + 1))$,
- for all $(i, j), (i + 1, j) \in [0, k] \times [0, 2^n - 1]$ with $i < k$, $up(f(i, j)) = down(f(i + 1, j))$,
- $f(0, 0) = d_{init}$,
- $f(k, 2^n - 1) = d_{final}$.

The set of atomic proposition $\mathcal{A}$ is defined as follows:

$$\mathcal{A} = \mathcal{A}_{main} \cup \mathcal{A}_{tag} \cup \{u, s\}$$

where $f(m, n) = 7 + 13 + 7m^2 + wn^2 + n^3$

- $\mathcal{A}_{main} = \Delta \cup \{\$\} \cup \mathcal{A}_{num}$
- $\mathcal{A}_{num} = [1, n] \times \{0, 1\}$
- $\mathcal{A}_{tag} = \{\bar{t}, \bar{t}_1, \ldots, \bar{t}_{f(m,n)}, t_1, \ldots, t_{43+20n^2+11m^2+40n+4n^3}\}$

The main atoms in $\mathcal{A}_{main}$ are used to encode *cells*. A cells with content $d \in \Delta$ and column number $j \in [0, 2n - 1]$ is encoded by finite words in

$$cell_j := (1, b_1)^{+}(2, b_2)^{+} \ldots (2^n - 1, b_{2^n - 1})^{+}(d)^{+}$$

where $b_1, \ldots, b_n$ is the binary encoding of the column number j. Given the representation of a j-th cell, we can introduce how we encode the i-th row:

$$row_i := \{\$\}^{+}\ cell_0\ cell_1\ \ldots cell_{2^n - 1}$$

A tiling f is encoded by finite words w over $\mathcal{A}_{main}$ where w corresponds to a sequence of row encodings separated one from each other by $\{\$\}^+$.

$$w := \{\$\}^+ \, row_0 \, \ldots \{\$\}^+ \, row_k$$

The extra symbols in $\mathcal{A}_{tag}$ and $\{u, s\}$ have some special use to check if the prefix of the trace (projected over $\mathcal{A}_{main}$) contains a prefix that encodes a tiling. Intuitively, elements in $\mathcal{A}_{tag}$ are used to mark segments of the trace that should contain specific sets of atoms in $\mathcal{A}_{main}$, while u is used to simulate *ASP-negative constraints* such as $\neg\varphi \to \bot$, while s is used to simulate *positive constraints* such as $\varphi \to \bot$.

We recall a revisited definition of the *pseudo-tiling* as defined in (Bozzelli and Pearce 2016) for the $\mathrm{THT}_2^1(\Box, \Diamond)$ fragment. An interpretation $\langle \mathbf{H}, \mathbf{T} \rangle$ over $\mathcal{A}$ is a pseudo-tiling if:

(I) for each $i \geq 0$ such that $|\mathcal{A}_{main} \cap H_i| \geq 1$,

(II) $\$ \in H_0$,

(III) there is $i \geq 0$ such that $d_{final} \in H_i$,

(IV) 1. either for all $i \geq 0$ $u \in T_i$ and $T_i \cap \mathcal{A}_{tag} = \mathcal{A}_{tag}$, or
2. $u \notin T_0$ and for all $i \geq 0$ $\bar{t} \notin T_i$ and $|T_i \cap \mathcal{A}_{tag}| \geq 1$,

(V) if $u \notin H_0$, then $|H_i \cap \mathcal{A}_{tag}| \geq 1$ and $\bar{t} \notin H_i$ for all $i \geq 0$,

(VI) if there exists an $i \geq 0$ such that $|\mathcal{A}_{main} \cap \mathbf{H}_i| > 1$, then we force infinitely many s in $\mathbf{H}$,

(VII) there must be infinitely many u in $\mathbf{H}$ and, finally

(VIII) $\bar{t} \in H_i$ iff $\bar{t}_j \in H_{i_j}$ for each $j \in \{1, \ldots, f(m,n)\}$ for some i_j.

Furthermore, if a pseudo-tiling satisfies $u \in T_0$, then it is called *good*. The main difference with respect to the original definition is the use of s as a constraint since whenever we have more the one main atom in a state, we must satisfy infinitely many s, but we can always remove some of them and still have infinitely many s in the trace.

We can make the following observation:

Observation 1 (Not good traces) *Let $\mathbf{T}$ be a total pseudo-tiling code for $\mathrm{GP}_2(\Box, \Diamond)$ which is not good. Then, $u \notin T_0$ in $\mathbf{T}$, and there exists $\mathbf{H}$ such that $\mathbf{H} < \mathbf{T}$, $\langle \mathbf{H}, \mathbf{T} \rangle$ is a pseudo-tiling code, which coincides with $\mathbf{T}$ on $\mathcal{A} \setminus \{u, \bar{t}\}$, but where some u and all the $\bar{t}$ have been removed.*

We can construct in polynomial time an $\mathrm{GP}_2(\Diamond, \Box)$ formula φ_{pseudo} such that $\langle \mathbf{H}, \mathbf{T} \rangle \vDash \varphi_{pseudo}$ iff $\langle \mathbf{H}, \mathbf{T} \rangle$ is a pseudo-tiling code for $\mathrm{GP}_2(\Diamond, \Box)$.

$$\underbrace{\Box(\Diamond u)}_{\text{(VII)}} \wedge \underbrace{\{\$\}}_{\text{(II)}} \wedge \underbrace{\{\Diamond d_{final}\}}_{\text{(III)}} \wedge \underbrace{\Box(\bigvee_{p \in \mathcal{A}_{main}} p)}_{\text{(I)}}$$

$$\text{(IV),(V),(VI)} \begin{cases} \Box(\displaystyle\bigvee_{p \in \mathcal{A}_{tag} \setminus \{\bar{t}, \bar{t}_0, \ldots, \bar{t}_8\}} p) \wedge \\ (\displaystyle\bigvee_{p \in \mathcal{A}_{main}} \Diamond(p \wedge \bigwedge_{p' \in \mathcal{A}_{main} \setminus \{p\}} p')) \to \Box\Diamond s \wedge \\ [u \vee \Diamond(\bar{t})] \to \Box(u \wedge \displaystyle\bigwedge_{p \in \mathcal{A}_{tag}} p) \end{cases}$$

$$\underbrace{\Diamond\bar{t} \leftrightarrow (\Diamond\bar{t}_1 \wedge \Diamond\bar{t}_{f(m,n)})}_{\text{(VIII)}}$$

We introduce a template that allows us to derive infinitely many s when we recognize incorrect patterns in the pseudo-tiling. Let $t_{i_1}, \ldots, t_{i_k} \in \mathcal{A}_{tag}$, and $\emptyset \subset \mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{A}_{main}$. Then, one can construct in polynomial time a $\mathrm{GP}_2(\Diamond, \Box)$ formula $\theta(i_1 \mid P_1, \ldots, i_k \mid P_k)$ over $P \setminus \{u, s\}$ s.t. for all good pseudo-tiling codes $\langle \mathbf{H}, \mathbf{T} \rangle$ for $\mathrm{GP}_2(\Diamond, \Box)$ with $u \notin H_0$,

$$\langle \mathbf{H}, \mathbf{T} \rangle \models \theta_l(i_1 \mid P_1, \ldots, i_k \mid P_k) \text{ iff}$$

the projection of $\mathbf{H}$ over $\mathcal{A}_{tag}$ is in $\{t_{i_1}^+\} \ldots \{t_{i_{t-1}}^+\}\{t_{i_t}^\omega\}$, for all $1 \leq j \leq k$, all the main propositions which label the segment of $\mathbf{H}$ marked by t_{i_j} are in P_j, and there are infinitely many s in $\mathbf{H}$.

For a good pseudo tiling code $\langle \mathbf{H}, \mathbf{T} \rangle$ with $u \notin H_0$, $T_i \cap \mathcal{A}_{tag} = \mathcal{A}_{tag}$ and $\bar{t} \notin H_i$ for all $i \geq 0$. Hence, $\langle \mathbf{H}, \mathbf{T} \rangle \not\models \Diamond\bar{t}$. $\theta_l(i_1 \mid P_1, \ldots, i_k \mid P_k)$ can be formulated as

$$\Box\Diamond s \text{ are derived } (\bigwedge_{j \in [1,\ldots,k]} \Diamond t_{i_j}) \to \Box\Diamond s$$

$$p \notin P_j \text{ for } i_j \; (\bigvee_{j \in [1,\ldots,k]} \bigvee_{p \in \mathcal{A} \setminus P_j} \Diamond(t_{i_j} \wedge p)) \to \Diamond\bar{t}_l$$

$$\text{incorrect } i_j \text{ order } (\bigvee_{r,r' \in [1,\ldots,k]:r<r'} \Diamond(t_{i_{r'}} \wedge \Diamond(t_{i_r}))) \to \Diamond\bar{t}_l$$

One can construct in polynomial time a $\mathrm{GP}_2(\Box, \Diamond)$ formula φ_{unstab} over $\mathcal{A} \setminus \{u\}$ such that for all total interpretations $\mathbf{T}$ which are good pseudo-tiling codes for $\mathrm{GP}_2(\Box, \Diamond)$, there exists a good pseudo-tiling code for $\mathrm{GP}_2(\Box, \Diamond)$ of the form $\langle \mathbf{H}, \mathbf{T} \rangle$ with $\mathbf{H} \neq \mathbf{T}$ and satisfying φ_{unstab} iff there is no prefix of $\mathbf{T}$ whose projection over $\mathcal{A}_{main}$ encodes a tiling. We list out some properties that we want to fulfill to obtain a tiling out of a good pseudo-tiling:

1. The content of the first cell is not d_{init};
2. Either some \$-position is preceded by an incomplete prefix and is followed by a Δ-position, or some $\mathcal{A}_{num}$-position is preceded by an incomplete prefix and is followed by a \$-position;
3. No cell preceded by an incomplete prefix has content d_{final} and is the last cell of a row;
4. There are segments in $(\{\$\} \cup \Delta \setminus \{d_{final}\})\mathcal{A}_{num}^+\Delta^+$, preceded by incomplete prefixes, such that the suffix in $\mathcal{A}_{num}^+\Delta^+$ is not a correct encoding of a cell;
5. There is a row preceded by an incomplete prefix whose first (resp. last) cell has column number distinct from 0 (resp. $2^n - 1$);
6. There are adjacent cells in a row, preceded by an incomplete prefix, whose column numbers are not consecutive;
7. Bad row (resp. column) condition: there are two adjacent cells in a two (resp. column), preceded by an incomplete prefix, which have different color on the shared edge;
8. There is no position labeled by d_{final} that follows an incomplete prefix.

We show how to encode point 1. of the previous requirements referencing to (Bozzelli and Pearce 2016) for the encoding of the others. A key difference with respect to the original proof for $\mathrm{THT}^1_2(\Diamond, \Box)$ is that all the θ_i must be put in conjunction and all the markers t_i cannot be reused, this is the reason why we need so many elements in $\mathcal{A}_{num}$, but still polynomial in the size of the input.

$$\varphi_1 := \theta_1(1 \mid \{\$\}, 2 \mid \mathcal{A}_{num}, 3 \mid \Delta \setminus \{d_{init}\}, 4 \mid \mathcal{A}_{main})$$

$$\varphi_2 := \begin{cases} \theta_2(5 \mid \mathcal{A}_{main}, 6 \mid \$, 7 \mid \Delta, 8 \mid \mathcal{A}_{main}) \wedge \\ \theta_3(9 \mid \{\$\}, 10 \mid \Delta, 11 \mid \mathcal{A}_{main}) \wedge \\ \theta_4(12 \mid R_{main}, 13 \mid \mathcal{A}_{num}, 14 \mid \$, 15 \mid \mathcal{A}_{main}) \end{cases}$$

φ_7 can be obtained putting in a conjunction for all $(d, d') \in (\Delta \setminus \{d_{final}\} \times \Delta : right(d) \neq left(d')$ with $i := (d, d')$ the formulas

$$\theta(\mathcal{A}_{main}, 7_{2,i} \mid \{d\}, 7_{3,i} \mid \mathcal{A}_{num}, 7_{4,i} \mid \{d'\}, 7_{5,i} \mid \mathcal{A}_{main})$$

The remaining condition can be easily obtained from (Bozzelli and Pearce 2016) following the above suggested rewriting for conditions 1, 2, and 7.

Therefore, we define

$$\varphi_{unstab} := \theta_1(\dots) \wedge \dots \wedge \theta_{f(m,n)}(\dots).$$

One can construct in polynomial time a $\mathrm{GP}_2(\Box, \Diamond)$ formula $\varphi_{\mathcal{I}}$ s.t. there is an equilibrium model of $\varphi_{\mathcal{I}}$ iff there is a tiling of $\mathcal{I}$.

Let φ_{pseudo} and φ_{unstab} the two $\mathrm{GP}_2(\Box, \Diamond)$ above defined. Then,

$$\varphi_{\mathcal{I}} := \varphi_{pseudo} \wedge \varphi_{unstab} \wedge (u \vee \Box\Diamond s)$$

Let us argue that the construction is sound. Let us assume that $\mathbf{T}$ is an equilibrium model of $\varphi_{\mathcal{I}}$, then we know that it is a good pseudo-tiling. It is a pseudo-tiling code because it satisfies φ_{pseudo} and it is good because there are infinitely many u in $\mathbf{H}$, therefore in order to get stability we need to derive all of them, and the only rule that can be used is the last one in the definition of φ_{pseudo}, which makes u true throughout all the trace, see Observation 1. Furthermore, all the conditions for a pseudo-tiling code to represent a tiling code must be fulfilled, otherwise, if one in (1.-8.) was not fulfilled or if there were more than one atom in $\mathcal{A}_{main}$ in at least one state, we would have had infinitely many s that would have caused instability.

Let us show that the construction is complete. Assume that there exists a tiling f, then let $\mathbf{T}$ be any good pseudo-tiling code such that the projection of some prefix of $\mathbf{T}$ over $\mathcal{A}_{main}$ is an encoding of f. Since $u \in T_0$, then $\mathbf{T} \models \varphi_{\mathcal{I}}$. Let us assume that there exists $\mathbf{H} \subset \mathbf{T}$ such that $\langle \mathbf{H}, \mathbf{T} \rangle \models \varphi_{\mathcal{I}}$. Since $\mathbf{H} \neq \mathbf{T}$ and $\mathbf{T}$ is a good pseudo-tiling, then $\mathbf{H}$ and $\mathbf{T}$ can differ only on u and s. If $u \notin H_0$, then there are infinitely many s in $\mathbf{H}$. Such a situation happens if there are either more than one main atom in a state, which contradicts our hypothesis, or if one of the conditions (1.-8.) of $\mathbf{T}$ being an encoding of a tiling is violated, which would imply that f is not a tiling. □

Lemma 10 (Finite stable models characterization for THT^1) *Let Γ be a theory in THT^1. Minimal LTL models of Γ coincide with stable models of Γ. Furthermore, the existence of a finite LTL model for Γ, implies the existence of a minimal one.*

Proof. Given a Γ theory, we know that if $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle \models \Gamma$, then $\mathbf{H} \models \Gamma$ because of Lemma 8.

Therefore, let $\mathbf{T}$ be a minimal LTL model and assume for the sake of contradiction that there exists $\mathbf{H} < \mathbf{T}$ such that $\langle \mathbf{H}, \mathbf{T} \rangle \models \Gamma$, then $\mathbf{H}$ would be an LTL model smaller than $\mathbf{T}$ contradiction the hypothesis.

Let us assume that $\mathbf{T}$ is a stable model of Γ, then it is also an LTL model of Γ. If it is not already a minimal LTL model of Γ, then by the finiteness of $\mathbf{T}$ we can remove atoms from $\mathbf{T}$ and obtain a minimal $\mathbf{H}$ LTL model of Γ. This second point proves the second statement of the Lemma. □

Proof of Theorem 7.

- The proof for the GP-hardness result is given in Lemma 9.
- The hardness for the $\mathrm{THT}^2_2(\mathsf{U})$ fragment satisfiability follows directly from the construction φ given in (Bozzelli and Pearce 2016).
- The NP completeness for the $\mathrm{THT}^1_2(\Box, \Diamond)$ fragment follows from Lemma 10 and a result in (Fionda and Greco 2016). Furthermore, if we add the formula $\Box\Diamond p$ to the theory, where p is a fresh atom, then we can admit also the F operator, which is rewritten into p, if it appears in the body of the rules because p can be true only at the final state. If it appears in the head of a rule, then we replace it with u and we add $\Box(u \vee \bar{u})$, $\Box(u \wedge \bar{u} \to \bot)$, $\Box(p \wedge \bar{u} \to \bot)$. So that u occurs only when p occurs, namely at the last state of the trace.
- The PSPACE-completeness for the$\mathrm{THT}^1_2(\Box, \Diamond, \circ)$ fragment follows from Lemma 10 and a result in (Fionda and Greco 2016). Furthermore, if we add the formula $\Diamond p \wedge \Box(\circ\top \wedge p \to bot)$ to the theory, we can replace every occurrence of the F operator in the theory; moreover, we can also replace each occurrence of the $\widehat{\circ}(\varphi)$ formulas with $\circ(\varphi) \vee p$. The reason is that there must be a state i where $u \in H_i$ for each possible $\mathbf{M}$ THT model of Γ, and because of the constrain $\Box(\circ\top \wedge p \to bot)$, all the models must be of length $i + 1$.

□

Let us recall the definition of temporal programs for finite traces given in (Cabalar and Schaub 2019). The *previous* operator, not considered in this work, is denoted by $\bullet$. A temporal program is defined as follows.

Definition 13 (Temporal program) *Given an alphabet $\mathcal{A}$, we define the set of* temporal literals *as* $\{p, \neg p, \bullet p, \neg \bullet p \mid p \in \mathcal{A}\}$. *A* temporal program *is any set of* temporal rules *of one of the following forms:*

- *(*initial *rule)*
$$r : B \to A \qquad (20)$$
- *(*dynamic *rule)* $\widehat{\circ}\Box r$*, where r is an initial rule;*
- *(*final *rule) either* $\Box(\mathsf{F} \to r)$.

where $B = b_1 \wedge \cdots \wedge b_n$ with $n \geq 0$, $A = a_1 \vee \cdots \vee a_m$ with $m \geq 0$ and the b_i and a_j are temporal literals for dynamic rules, or regular literals $a, \neq a | a \in \mathcal{A}$ for initial and final rules. We denote by $B^+(r)$ (resp. $B^-(r)$) the set of (resp. negated) temporal atoms in the body of r, and by $H(r)$ the set $\{c_1, \ldots, c_l\}$.

Since we do not want to consider past operators, we can rewrite dynamic rules

$$\widehat{\circ}\Box(\bullet p \wedge q \rightarrow r)$$

into

$$\Box(p \wedge \circ q \rightarrow \circ r) \vee \mathbf{F}$$

Observation 2 *Deciding whether a $\neg$-free temporal programs as defined in (Cabalar and Schaub 2019) admits a finite stable model is PSPACE-c, while if it is also $\circ$-free then it is NP-c.*

Proof. Given the suggested rewriting of temporal programs for finite traces and the results from Theorem 7 we can conclude that the satisfiability problem for $\neg$-free temporal programs over finite traces is PSPACE complete and NP complete for $\neg$-free and $\circ$-free temporal. $\square$

Proof of Lemma 2. Given a $\mathrm{THT}^1_{m+1}(\Diamond, \Box)$ theory Γ, we know by Theorem III.1 in (Bozzelli and Pearce 2016) that TEL satisfiability is EXPSPACE-hard. But we also notice that formulas in Γ have implication height one and that are $\circ$-free. Therefore, we can use Theorem 3, from which we reduce the problem of c-stable model existence to the stable model existence. Furthermore, once we have the stable model, we can obtain the c-stable one just by removing stuttering.

A similar argument can be obtained for the THT^0 fragment while resorting to Theorem IV.1 in (Bozzelli and Pearce 2016).

The hardness for $\mathrm{THT}^2_2(\mathbf{U})$ is obtained by adding $\bigwedge_{p\in\mathcal{A}} (\neg p \vee \neg\neg p)\ \mathbf{U} \bigwedge_{p\in\mathcal{A}} (\neg p)$ to the construction proposed in (Bozzelli and Pearce 2016). The reason is that we do not admit defeaters different from T-stuttering ones. Therefore, we can appeal to 6, since after the first empty state there can be only empty sets, i.e. an *empty suffix*.

The PSPACE-hardness for $\mathrm{THT}^1(\circ, \mathbf{R})$ can be obtained by Theorem IV.4 in (Bozzelli and Pearce 2016) by imposing the non contraction axiom $\Box(\circ\top)$. $\square$

Proposition 23 *Deciding whether a given theory Γ has an c-stable model is in EXPSPACE.*

Proof.(Sketch) This can be shown using automata-theoretic techniques, similar as for the result that stable model existence is in EXPSPACE in general (Cabalar and Demri 2011). To this end, Cabalar and Demri roughly speaking constructed an NFA Büchi automaton A_φ that recognizes as a language (whose alphabet is the set of all interpretations of $\mathcal{A}$) the models of a given THT formula φ. A further NFA Büchi automaton $K(A)$ accepts all $\mathbf{T}$ that are not stable, witnessed by some $\mathbf{H}$ such that $\langle \mathbf{H}, \mathbf{T}\rangle \models \varphi$. The final automaton, which accepts the stable models of the input theory, intersects the language of A_φ, $\mathcal{L}(A)$ with the complement of the language $\mathcal{L}(K(A))$ of $K(A)$. The EXPSPACE upper bound is then derived using well-known results.

We can adapt this construction in order to check the condition $\langle \mathbf{T}', \mathbf{T}, \mu\rangle \models \varphi$ for general μ instead of the special case $\mu = id$. To this end, the representation of traces is modified, and the formula evaluation expressed in LTL.

Specifically, we represent $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu\rangle$ in a trace $\mathbf{T}_{\mathbf{M}}$, which uses for $\mathbf{T}'$ a copy $\mathcal{A}'$ of the variables $\mathcal{A}$ as in (Cabalar and Demri 2011), and a special variable m for marking in $\mathbf{T}_{\mathbf{M}}$ the positions $s_j = \min(\mu^{-1}(j))$ for all $j \geq 0$, i.e., the first position in the segment of $\mathbf{T}$ that is mapped to position j of $\mathbf{T}'$; note that $\mu = id$ holds iff each position in $\mathbf{T}_{\mathbf{M}}$ is marked. Furthermore, on all positions $s_j, s_{j+1}, \ldots$ in $\mu^{-1}(j)$, the value of $\mathbf{T}'_j$ is put.

For evaluating $\mathbf{T} \models \varphi$, we may take any Büchi automaton for evaluating φ under LTL semantics.

For evaluating $\mathbf{M} = \langle \mathbf{T}', \mathbf{T}, \mu\rangle \models \varphi$ where $\mathbf{T}' \neq \mathbf{T}$, we may express that $\langle \mathbf{T}', \mathbf{T}, \mu\rangle$ is proper and $\mathbf{T}' \neq \mathbf{T}$ in LTL, using formulas

$$m \wedge \Box(T' \leq T), \tag{21}$$

$$\Box(\widehat{\circ} m \vee T' = \widehat{\circ} T'), \tag{22}$$

$$\Diamond \bigvee\nolimits_{p\in\mathcal{A}} \neg p' \wedge p \tag{23}$$

The evaluation of formulas in $\langle \mathbf{T}', \mathbf{T}, \mu\rangle$ at $\mathbf{T}'$ resp. $\mathbf{T}$ can then be defined by a structural transformation $f_w(\varphi)$, where $w = \mathbf{T}', \mathbf{T}$, which mirrors the conditions of the definition, using the variables in $\mathcal{A}$ resp. $\mathcal{A}'$.

In more detail, $f_{\mathbf{T}}(\varphi) = \varphi$, while $f_{\mathbf{T}'}(\varphi)$ is defined by $f_{\mathbf{T}'}(\top) = \top$, $f_{\mathbf{T}'}(\bot) = \bot$, $f_{\mathbf{T}'}(p) = p'$, $f_{\mathbf{T}'}(\varphi_1 \otimes \varphi_2) = f_{\mathbf{T}'}(\varphi_1) \otimes f_{\mathbf{T}'}(\varphi_2)$ for $\otimes \in \{\wedge, \vee\}$, etc.; in particular for implication we have

$$f_{\mathbf{T}'}(\varphi_1 \rightarrow \varphi_2) = (f_{\mathbf{T}'}(\varphi_1) \rightarrow f_{\mathbf{T}'}(\varphi_2)) \wedge f_{\mathbf{T}}(\varphi_1 \rightarrow \varphi_2)$$

and for the $\circ$ operator

$$f_{\mathbf{T}'}(\circ\varphi) = m \wedge \circ(f_{\mathbf{T}'}(\varphi)). \tag{24}$$

Then $\langle \mathbf{T}', \mathbf{T}, \mu\rangle \models \varphi$ amounts to $\mathbf{T}'_{\mathbf{M}} \models f_{\mathbf{T}'}(\varphi)$.

In order to avoid an exponential blowup in the rewriting, we may use auxiliary symbols p_ψ resp. p'_ψ to name subformulas ψ, and require that $p_\psi \leftrightarrow f_{\mathbf{T}}(\psi)$ and $p'_\psi \leftrightarrow f_{\mathbf{T}'}(\psi)$, where the names are recursively used.

In summary, the automata-based argument in (Cabalar and Demri 2011) can be adapted to show that deciding c-stable model existence is feasible in EXPSPACE. $\square$

Proposition 24 *Deciding whether a given theory Γ has an c-stable model is EXPSPACE-complete in general.*

Proof.(Sketch) By Lemma 2 and Propositions 23, it remains to argue about finite c-stable models. We may enforce finiteness, without compromising c-stability, by adding the formula

$$in \wedge \Box((in \vee out) \wedge (in \wedge out \rightarrow \bot)) \wedge \Box(out \rightarrow (\Box out \wedge \Diamond out)). \tag{25}$$

Note that this formula is in GP and in THT_2^1.

Here out marks the area which is out of a simulated finite trace. The formulas in Γ are modified to restrict their scope to the area where in is true, by changing φ to $in \wedge \varphi$, and temporal subformulas $\varphi_1 \,\mathbf{R}\, \varphi_2$, $\varphi_1 \,\mathbf{U}\, \varphi_2$, $\circ\varphi$, and $\widehat{\circ}\varphi$ are adjusted to $(\varphi_1 \wedge in) \vee out \,\mathbf{R}\, (\varphi_2 \wedge in)$, $(\varphi_1 \wedge in) \,\mathbf{U}\, (\varphi_2 \wedge in)$, $\circ(in \wedge \varphi)$, and $\circ(in \wedge \varphi) \vee out$ respectively. In the area marked with out, no other atoms than out can be true in any stable model, thus creating a strongly ultimately periodic trace. The c-stability condition is inside the area marked with out not affected. That is, if $\mathbf{T}_1$ and $\mathbf{T}_2$ are traces such that $\langle \mathbf{T}_1, \mathbf{T}_2, \mu \rangle \models \Gamma$, then for the modified Γ' and the infinite traces $\mathbf{T}_1'$ and $\mathbf{T}_2'$ representing $\mathbf{T}_1$ and $\mathbf{T}_2$, respectively, we shall have $\langle \mathbf{T}_1', \mathbf{T}_2', \mu' \rangle \models \Gamma'$, where μ' extends μ to $[0, \infty)$ if needed. This also works in the converse direction. $\square$

Proof of Theorem 8. The result follows from Proposition 23 and Proposition 24. $\square$

Proof of Proposition 19. Any $\circ$-free THT_1^1 theory Γ has by Theorem 3 some c-stable model iff it has some stable model $\mathbf{T}$. The latter can by Lemma 1 be stuttered into an infinite model $\mathbf{T}'$; it is easy to see that $\mathbf{T}'$ is then also stable. It has been shown in (Bozzelli and Pearce 2015) that deciding whether Γ has an infinite stable model is NP-complete. This shows the result for THT_1^1.

For GP_1, we thus obtain NP-membership as it is a subclass of THT_1^1. The NP-hardness follows immediately from deciding whether a given positive propositional disjunctive logic program P, with empty rule heads allowed, has some stable mode, which is known to be NP-complete (Eiter and Gottlob 1995). $\square$

8 Semantic Properties

In LTL and THT with only future operators, we can always transform satisfaction of a trace $\mathbf{M}$ at point k into satisfaction of a suffix of the trace at 0.

Definition 14 (Trace shifting) *Given a* cTHT*-trace* $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu \rangle$ *with* $\lambda = |\mathbf{H}|$ *and* $\lambda' = |\mathbf{T}|$ *we define the* shifting *of* $\mathbf{M}$ *by a distance* $d \in [0..\lambda)$*, denoted by* $\mathbf{M}[d]$*, as the new* cTHT*-trace* $\mathbf{M}[d] = \langle \mathbf{H}', \mathbf{T}', \mu' \rangle$ *where:*

- *we define* $d' =_{\text{def}} \Sigma_{i=0}^{d-1} |\mu^-(i)|$
- μ' *is a contractor from* $\lambda' - d'$ *to* $\lambda - d$ *(we assume* $\omega - n = \omega$ *for any natural number* n*) satisfying* $\mu(j - d') = \mu(j) - d$ *for all* $j \in [0..\lambda' - d')$.
- $\mathbf{T}' = (T'_i)_{[0..\lambda'-d')}$ *with* $T'_i = T_{i+d'}$
- $\mathbf{H}' = (H'_i)_{[0..\lambda-d)}$ *with* $H'_i = H_{i+d}$

The following property can be easily observed by a simple analysis of THT satisfaction conditions:

Proposition 25 $\mathbf{M}, k \models \varphi$ *iff* $\mathbf{M}[k], 0 \models \varphi$.

It is interesting to note that some unfolding features of the $\mathbf{U}$ and $\mathbf{R}$ operators from THT and LTL do not hold anymore in cTHT. Consider the following THT-tautology

$$\varphi \,\mathbf{U}\, \psi \leftrightarrow (\psi \vee (\varphi \wedge \circ(\varphi \,\mathbf{U}\, \psi)))$$

It is possible to see that $\mathbf{M} = \langle \mathbf{H}, \mathbf{T}, \mu \rangle$ where $\mathbf{T} = \{p\} \cdot \{p\} \cdot \{q\} \cdot$, $\mathbf{H} = \{p\} \cdot \{q\} \cdot$, $\mu^-(0) = \{0, 1\}$, and $\mu^-(1) = \{2\}$ is a model of $\varphi \,\mathbf{U}\, \psi$, but not of $\psi \vee (\varphi \wedge \circ(\varphi \,\mathbf{U}\, \psi))$. For a similar reason the following formula is not a tautology in cTHT.

$$\varphi \,\mathbf{R}\, \psi \leftrightarrow (\psi \wedge (\varphi \vee \circ(\varphi \,\mathbf{R}\, \psi)))$$

However, in the next Lemma, we ensure that under some conditions on the trace, we can lift from the THT case some unfolding properties of the temporal operators to the cTHT case.

Lemma 11 (Integral unfolding) *Given a* cTHT *trace* $\mathbf{M}$ *such that the* $i \in [0, \ldots \lambda)$ *state is integral, i.e.* $|\mu^-(i)| = 1$*, then the following equivalences hold*

(a) $\mathbf{M}, i \models \circ\varphi$ *iff* $i + 1 < \lambda$ *and* $\mathbf{M}, i + 1 \models \varphi$*,*

(b) $\mathbf{M}, i \models \varphi \,\mathbf{U}\, \psi$ *iff* $\mathbf{M}, i \models \psi \vee (\varphi \wedge \circ(\varphi \,\mathbf{U}\, \psi))$*, and*

(c) $\mathbf{M}, i \models \varphi \,\mathbf{R}\, \psi$ *iff* $\mathbf{M}, i \models \psi \wedge (\varphi \vee \circ(\varphi \,\mathbf{R}\, \psi))$.

Proof. We can prove all the points by induction on the structural complexity of the formula φ. The propositional case is straightforward. Let us analyze what happens with $\circ\varphi$, case (a). By Defn. 7 we know that

$$\mathbf{M}, i \models \circ\varphi \iff |\mu^-(i)| = 1,\ i{+}1 < \lambda,\ \text{and } \mathbf{M}, i{+}1 \models \varphi$$

since by assumption $|\mu^-(i)| = 1$, we only require $\mathbf{M}, i{+}1 \models \varphi$ to hold. We give the detailed proof for $\varphi \,\mathbf{U}\, \psi$ case (b), but we leave out the on for case (c) since it has a similar argument of (b).

Let us assume that $i{+}1 < \lambda$. By Defn. 7 we know that

$$\mathbf{M}, i \models \varphi \;\mathbf{U}\; \psi \iff \begin{cases} \text{for some } j \in [i..\lambda), \text{ we have } \mathbf{M}, j \models \psi \\ \text{and } \mathbf{M}, k \models \varphi \text{ for all } k \in [i..j) \end{cases}$$

$$\iff \begin{cases} \text{either (1) } \mathbf{M}, i \models \psi, \text{ or (2)} \\ \text{for some } j \in [i{+}1..\lambda), \text{ we have } \mathbf{M}, j \models \psi \\ \mathbf{M}, i \models \varphi \text{ and} \\ \mathbf{M}, k \models \varphi \text{ for all } k \in [i{+}1..j) \end{cases}$$

Exploiting point (a) and the boolean connective cases, we obtained the desired rewriting, since either ψ holds at the state i or φ holds at i and we have an $\mathbf{U}$ formula to satisfy at position $i{+}1$.

The $i{+}1 = \lambda$ case is simpler, since the only option for j in the Defn. 7 is $j = i$, implying that the first argument of the $\mathbf{U}$ does not play any rule. Therefore the rewriting holds in the case as well, since the second disjunction, namely $(\varphi \wedge \circ(\varphi \,\mathbf{U}\, \psi))$ cannot be evaluated true, since the next operator does not hold in the last position. $\square$

We can identify interesting tautologies that involve the next operator too.

- two ways to express synchronicity: $\Box(\Box\hat{\circ}\top \leftrightarrow (\Box\circ\top) \vee (\circ\top \,\mathbf{U}\, \neg\circ\top))$

9 Alternative Definitions of A-Stable Models

Above, we have defined a-stable models by requiring for an LTL-model $\mathbf{T}$ that

$$\forall \mathbf{H}, \mu : \langle \mathbf{H}, \mathbf{T}, \mu \rangle \models \Gamma \text{ implies } \mathbf{H} = \mathbf{T} \qquad \text{(A1)}$$

We might consider alternative conditions:

$$\forall \mathbf{H}, \mu : \langle \mathbf{H}, \mathbf{T}, \mu \rangle \models \Gamma \text{ implies } \langle \mathbf{T}, \mathbf{H}, id \rangle \models \Gamma \qquad \text{(A2)}$$

$$\forall \mathbf{H}, \mu : \langle \mathbf{H}, \mathbf{T}, \mu \rangle \models \Gamma \text{ implies } \exists \mu' \langle \mathbf{T}, \mathbf{H}, \mu' \rangle \models \Gamma \quad \text{(A3)}$$

Informally, (A2) allows that $\mathbf{T}$ is a componentwise smaller trace than $\mathbf{H}$, while (A3) says that we must be able to summarize $\mathbf{H}$ into $\mathbf{T}$; note that by Persistence, in both cases $\mathbf{H}$ must then be an LTL-model; i.e., if $\mathbf{H}$ is not an LTL-model and $\langle \mathbf{H}, \mathbf{T}, \mu \rangle \models \Gamma$, then $\mathbf{T}$ is not a-stable.

The three conditions (A1)-(A3) are progressively more permissive, meaning that stability under condition (Ai) implies stability under (Aj), $j \geq i$; the converse is not true, i.e., the conditions are strictly more permissive.

9.1 A2 (vs) A3

To show this for (A2) and (A3), we use the following example.

Example 10 *Consider*

$$\Gamma = \{\neg p,\ \circ\top \to \circ\circ p,\ \Diamond p,\ \Box(p \to \circ p)\}.$$

Then the only s-stable model is $\mathbf{T} = \emptyset \cdot \emptyset \cdot \{p\}^\omega$*, as we must have* p *at* $i = 2$ *and there is no need to put* p *at* $i = 1$ *(it must not be at* $i = 0$*). At* $i \geq 3$*, we then always must have* p*.*

According to (A1), $\mathbf{T}$ *is not a-stable, as we can map it to the trace* $\mathbf{H} = \emptyset \cdot \{p\}^\omega$*, by contracting* T_0 *and* T_1 *with* μ*, and then* $\langle \mathbf{H}, \mathbf{T}, \mu \rangle \models \Gamma$ *holds.*

Let $\mathbf{T}_i = \emptyset^i \cdot p^\omega$*,* $i \geq 1$*; then* $\mathbf{T} = \mathbf{T}_2$ *and* $\mathbf{H} = \mathbf{T}_1$*.*

We examine when $\mathbf{T}_i, \mathbf{T}_j, \mu \models \Gamma$*,* $i, j \geq 1$*, is possible, i.e., some* μ *exists:*

- *as $\circ\top \rightarrow \circ\circ p$ must be satisfied at position* 0 *in* $\mathbf{T}_j$*, we must have that p is true at position 2 of* $\mathbf{T}_j$*, and* $j \leq 2$ *holds.*
- *If $\mu(1) = 1$, then $\circ\circ p$ must be satisfied at position 0, thus $\mu(2) = 2$ must hold and p must be true at position 2 of* $\mathbf{T}_i$*, i.e.,* $i \leq 2$.
- *If $\mu(1) = 0$, then $j \geq 2$, hence $j = 2$ must hold, as otherwise $\neg p$ can not be satisfied at position 0. On the other hand, i can be arbitrary: we can set $\mu(0) = 0$ and $\mu(k) = k - 1$, $k \geq 1$, as from some position on in $\mathbf{T}_i$ p is always satisfied.*

Consequently, under (A2), $\mathbf{T} = \mathbf{T}_2$ *is not stable, as e.g.* $\langle \mathbf{T}_3, \mathbf{T}_2, \mu \rangle \models \Gamma$ *is possible but* $\langle \mathbf{T}_2, \mathbf{T}_3, id \rangle \models \Gamma$ *does not hold (as as* $\mathbf{T}_3 \not\downarrow^{id} \mathbf{T}_2$*); similarly, also* $\mathbf{T} = \mathbf{T}_1$ *is not stable, as for* $\langle \mathbf{T}_2, \mathbf{T}_1, \mu \rangle \models \Gamma$ *we have that* $\langle \mathbf{T}_1, \mathbf{T}_2, id \rangle \models \Gamma$ *does not hold.*

Under (A3), $\mathbf{T} = \mathbf{T}_2$ *is not stable, as e.g.* $\langle \mathbf{T}_3, \mathbf{T}_2, \mu \rangle \models \Gamma$ *is possible but not* $\langle \mathbf{T}_2, \mathbf{T}_3, \mu' \rangle \models \Gamma$ *(as* $\mathbf{T}_3$ *is not an* LTL*-model of* Γ*). However,* $\mathbf{T} = \mathbf{T}_1$ *is stable, as for* $\langle \mathbf{T}_i, \mathbf{T}_1, \mu \rangle \models \Gamma$ *we have* $i = 1, 2$ *and* $\langle \mathbf{T}_1, \mathbf{T}_2, \mu' \rangle \models \Gamma$ *is possible. Noticeable,* $\mathbf{T}'$ *is not an s-stable model of* Γ*.* $\square$

9.2 A1 (vs) A2

That (A1) strictly implies (A2) is shown by a more involving example.

Example 11 *Consider* $\Gamma = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$*, where*

$$\begin{aligned}
\varphi_1 &= \neg p, \\
\varphi_2 &= \Box(p \rightarrow \circ p), \\
\varphi_3 &= \circ\top \rightarrow \circ\circ p \\
\varphi_4 &= \Diamond p \vee \Box\circ\top
\end{aligned}$$

The only two possible LTL*-models of* Γ *are* $\mathbf{T} = \emptyset{\cdot}\emptyset{\cdot}(\{p\})^\omega$ *and* $\mathbf{H} = \emptyset{\cdot}(\{p\})^\omega$*. Roughly speaking, p cannot be true in position 1 and it must be true somewhere in the future because of $\varphi_4 \wedge \varphi_3$. If the first disjunction of φ_4 is satisfied we must have p somewhere in the future. Otherwise, if we have an infinite trace, then there is a next state and we trigger the implication in φ_3. Either way, we trigger the outer implication φ_3, which implies that p occurs already at the second state or at the third. Once we see it in the trace, we propagate p because of φ_2.*

We claim that $\mathbf{H}$ *is not an a-stable model under (A1) nor (A2), while* $\mathbf{T}$ *is a-stable under (A2), but not under (A1).*

If we consider $\mathbf{H}$ *as a candidate, then we soon realize that it cannot be an a-stable model either under (A1) or (A2) because it is not even an s-stable model.*

If we consider $\mathbf{T}$ *as a candidate, we can contract it into* $\mathbf{H}$ *with a contractor μ such that $\mu^-(0) = [0, 1]$ $\mu^-(i) = i + 1$ for $i \geq 1$ would be a a-THT model, i.e.,* $\langle \mathbf{H}, \mathbf{T}, \mu \rangle \models \Gamma$*. To see it, we can easily check that φ_i for $i = 1, 2, 3, 4$ are satisfied. Regarding φ_3 we notice that in the Here trace the implication is not triggered, since the first step is asynchronous, but it has to hold also in the There trace for each $i \in \mu^{-1}(0)$. At position 0,* $\langle \mathbf{T}, \mathbf{T}, id \rangle, 0 \models \circ\circ p$ *since $p \in T_2$.*

However, we also have that $\langle \mathbf{T}, \mathbf{H}, id \rangle \models \Gamma$*, which is also the reason why* $\mathbf{H}$ *is not an s-stable model. φ_1 is satisfied since both H_0 and T_0 are empty. From $i \geq 2$, $p \in T_i$, therefore φ_2 and φ_3 are satisfied too. We also notice that* $\langle \mathbf{H}, \mathbf{T}, id \rangle, i \models \circ\circ p$*, which imlpies that* $\langle \mathbf{H}, \mathbf{T}, id \rangle \models \varphi_4$.

$\mathbf{H}$ *is the unique non-trivial summarization of* $\mathbf{T}$ *becuase the summarization must be an* LTL *model, and the unique two* LTL *models are* $\mathbf{T}$ *and* $\mathbf{H}$. $\square$

9.3 Syntactic relaxation, so far A1 < A2 (?) A2' < A3 < A3'

A variant of (A2) and (A3) is by merely requiring that $\mathbf{H} \downarrow^{id} \mathbf{T}$ resp. $\mathbf{H} \downarrow^{\mu'} \mathbf{T}$ for some μ' holds and dropping $\langle \mathbf{T}, \mathbf{H}, id \rangle \models \Gamma$ resp. $\langle \mathbf{T}, \mathbf{H}, \mu' \rangle \models \Gamma$. The resulting conditions (A2$'$) and (A3$'$) also yield a strict hierarchy, as Example 10 and 11 separate (A2$'$) and (A3) resp. (A1) and (A2$'$). However, we can separate (A1) and (A2$'$) by using a simpler theory

$$\Gamma = \{\neg p,\ \Box(\neg p \vee p),\ \Diamond p,\ \Box(p \rightarrow \circ p)\}.$$

The s-stable models of Γ are of the form $\mathbf{T}_i = \emptyset^i{\cdot}\{p\}^\omega$, $i \geq 1$, and the single a-stable model under (A1) is $\mathbf{T}_1$. Under (A2$'$), each $\mathbf{T}_i$ is a-stable as $\mathbf{H}, \mathbf{T}_i, \mu \models \Gamma$ is only possible if $\mathbf{H}$ is of the form $\mathbf{T}_j$ for $j \leq i$; but here $\mathbf{T}_j \downarrow^{id} \mathbf{T}_i$ holds.

With the following example, we want to separate A3 from A3'.

Example 12 (To be checked) *Consider the* LTL *traces* $\mathbf{T}_i = \emptyset^i \cdot \{p\}^\omega$ *with $i < 4$. They are the unique LTL models of the following formula φ:*

$$\circ\circ\circ p \wedge \Box(p \rightarrow \circ p) \tag{26}$$

$\mathbf{T}_3$ *is the unique s-stable and a-stable model under (A1), (A2), and (A3), since* $\langle \mathbf{T}_{i+1}, \mathbf{T}_i, id \rangle, 0 \models \varphi$ *and there is no mapping μ from* $\mathbf{T}_{i+1}$ *to* $\mathbf{T}_i$ *such that* $\langle \mathbf{T}_i, \mathbf{T}_{i+1}, \mu \rangle, 0 \models \varphi$ *for $0 \leq i \leq 2$. The reason is that the first three steps are required to be synchronous. However, it is possible to find a contraction such that* $\mathbf{T}_3 \downarrow^\mu \mathbf{T}_2$*, for instance with $\mu^{-1}(0) = \{0, 1\}$, and $\mu^{-1}(i) = \{i + 1\}$ for each $i \geq 1$. Therefore, under (A3'),* $\mathbf{T}_2$ *would be an a-stable model of φ.*

[Davide: A2 (vs) A2' still todo.]

10 Inconsistency and Compact Stable Models

Presence of the $\circ$-operator may cause that a theory lacks an a-stable model while some s-stable model exists, even if it is near to be a GP formula. We first show an example where the lack of summarization via an a-stable model is caused by the fact that a trace must be infinite, not to express either a complex pattern or a stuttering justified by synchronization, but to solve an instability problem along the trace.

Example 13 *Consider the theory*

$$\Gamma = \{\Box(\circ\top \rightarrow p),\ \Box(\neg p \rightarrow p)\}.$$

The single s-stable model of Γ is $\mathbf{T} = \{p\}^\omega$*. Indeed,* $\langle \mathbf{H}, \mathbf{T} \rangle \models \Box(\circ\top \rightarrow p)$ *implies that* $p \in \mathbf{H}_i$ *for every $i \geq 0$, thus* $\mathbf{H} = \mathbf{T}$ *must hold. However,* $\mathbf{T}$ *is not a-stable since* $\mathbf{M} = \langle \emptyset, \{p\}^\omega, \mu \rangle$ *where $\mu(i) = 0$ for each $i \in \mathbb{N}$, fulfills* $\mathbf{M}, 0 \models \Gamma$*:* $\mathbf{M}, 0 \not\models \circ\top$ *and* $\mathbf{M}, 0 \not\models \neg p$*, while* $\mathbf{T}, \mu^-(0) \models p$*, which means that condition 5 of satisfaction* $\mathbf{M}, 0 \models \varphi \rightarrow \psi$ *is satisfied.*

The theory in the previous example has a single s-stable model that is infinite; the existence of finite s-stable models does not change the picture.

Example 14 *Consider the theory*

$$\Gamma = \{\neg\circ\top \rightarrow \circ\top\}.$$

Note that $\emptyset \cdot (\emptyset^+ + \emptyset^\omega)$ are all the s-stable models of the theory. However, none of them is a-stable because they can be summarized into an empty state.

As a possible alternative to a-stability, which aims at summarization of a model on the logical basis of HT, we may also consider a syntactic preference on s-stable models by reducing positive information with contractors, instead of reacquiring a justification for the trace length given by the dynamic of the behaviour of the trace.

Definition 15 (c-stable Model) *A trace $\mathbf{T}$ is a compact (c) stable model of a theory Γ, if $\mathbf{T}$ is an s-stable model of Γ and no s-stable model $\mathbf{T}'$ of Γ such that $\mathbf{T} \downarrow \mathbf{T}'$ and $\mathbf{T} \neq \mathbf{T}'$ exists.*

Revisiting the examples from above, in Example 13 the single s-stable model of Γ is trivially c-stable; in Example 14; $\mathbf{T} = \emptyset \cdot \emptyset$ is the single c-stable model of Γ, and in Example 6 (where a-stable models exist), $\mathbf{T}' = \{p\}$ is defeated by $\mathbf{T}'' = \emptyset$, which is a c-stable model. The latter also shows that c-stable models may eliminate a-stable models. However, this does not happen for $\circ$-free GP theories. As a consequence of Theorems 3 and 4, we obtain:

Proposition 26 *Let Γ be a set of $\circ$-free formulas. Then every c-stable model of Γ is an a-stable model of Γ, and if Γ consists of $\circ$-free* GP *formulas, then every a-stable model of Γ is also a c-stable model of Γ.*

However, c-stable models may not always exist when s-stable (or even a-stable) models exist, again for near-GP theories.

Example 15 *Consider the theory*

$$\Gamma = \{\neg p,\ \circ(p \vee \neg p),\ \circ\circ(p \wedge \Box(p \rightarrow \circ p))\}.$$

The s-stable models of Γ are $\mathbf{T}_1 = \emptyset\cdot\emptyset\cdot\{p\}^\omega$ and $\mathbf{T}_2 = \emptyset\cdot\{p\}^\omega$. As we have $\mathbf{T}_1 \downarrow^{id} \mathbf{T}_2$ and $\mathbf{T}_2 \downarrow^{\mu} \mathbf{T}_1$ for $\mu(0) = 0$, $\mu(i) = i - 1$, $i \geq 1$, none of them is c-stable. However, both $\mathbf{T}_1$ and $\mathbf{T}_2$ are a-stable models of Γ.

In the previous example, a more relaxed notion of c-stable model that requires $\mathbf{T}' \downarrow \mathbf{T}$ if $\mathbf{T} \downarrow \mathbf{T}'$ holds (rather than $\mathbf{T}' \neq \mathbf{T}$) would yield both $\mathbf{T}_1$ and $\mathbf{T}_2$ as c-stable models. However, this mutual-mapping relaxation does not guarantee us c-stable model existence in general.

Example 16 *Consider the theory*

$$\Gamma = \left\{ \begin{array}{l} \neg\circ\top \rightarrow \circ\top,\ (\circ\Box q_1) \rightarrow \bot,\ \circ\top \rightarrow \Box(\circ p_i \vee \circ q_i), \\ \Box(p_i \rightarrow p_{i+1}),\ \Box(q_{i+1} \rightarrow q_i),\ \Box(q_i \rightarrow \circ q_{i+1}),\ i \geq 1 \end{array} \right\}$$

The s-stable models of Γ are all infinite traces of the form $\mathbf{T}_i = \emptyset \cdot S_i \cdot S_{i+1} \cdot S_{i+2} \cdot \ldots$ where $S_i = \{q_j \mid j \in [1, i)\} \cup \{p_k \mid k \geq i\}$, for $i \geq 1$. Note that the first formula in Γ sensibly enforces a trace length greater than 1 (simply putting a formula $\circ\top$ would not work).

We can map each $\mathbf{T}_i$ to $\mathbf{T}_{i+1}$ using $\mu(0) = 0$ and $\mu(j) = j - 1$, for each $j \geq 1$, and thus $\mathbf{T}_{i+1} \preceq \mathbf{T}_i$ for each $i \geq 1$. On the other hand, we can not map $\mathbf{T}_{i+1}$ to any $\mathbf{T}_j$, $j \leq i$, as S_j does not occur in $\mathbf{T}_{i+1}$ and all sets S_i are pairwise incomparable; hence no surjective mapping μ from $\mathbf{T}_{i+1}$ to $\mathbf{T}_i$ is possible.

Thus we have the strict chain $\mathbf{T}_1 \succ \mathbf{T}_2 \succ \ldots$, which means that Γ has no c-stable model. Furthermore, $\langle \mathbf{T}_{i+1}, \mathbf{T}_i, \mu\rangle \models \Gamma$ holds (in particular, $\langle \mathbf{T}_{i+1}, \mathbf{T}_i, \mu\rangle \not\models \circ\top, \neg\circ\top$); hence Γ has no a-stable model either.

Furthermore, let us take $\mathbf{H} = \emptyset$ and $\mu(i) = 0$ for each $i \geq 0$. Then $\langle \mathbf{H}, \mathbf{T}_i, \mu\rangle \models \Gamma$ for each $i \geq 1$.

Under some conditions, c-stable models always exist if s-stable models exist. Let THT_1 denote the class of formulas with no nesting of temporal operators.

Proposition 27 *For every s-stable model $\mathbf{T}$ of a theory Γ of THT_1 formulas over a finite alphabet $\mathcal{A}$, there exists some c-stable model $\mathbf{T}' \preceq \mathbf{T}$ of Γ.*

Proof.(Sketch) Let S be the set of all s-stable models $\mathbf{T}'$ such that $\mathbf{T}' \preceq \mathbf{T}$ holds. We claim that S has a finite chain $\mathbf{T}_0 = \mathbf{T} \succ \mathbf{T}_1 \succ \cdots \succ \mathbf{T}_m$, $m \geq 0$, that can not be extended; hence $\mathbf{T}' = \mathbf{T}_m$ is a c-stable model such that $\mathbf{T}' \preceq \mathbf{T}$.

If S is a singleton, this is trivial. Otherwise, if some $\mathbf{T}' \in S$ has finite length, the claim holds since we can set $\mathbf{T}_1 = \mathbf{T}'$ and there is by the finiteness of $\mathcal{A}$ only finitely many $\mathbf{T}''$ such that $\mathbf{T}'' \prec \mathbf{T}'$ holds; thus by transitivity of $\preceq$, every chain as described must be finite. On the other hand, in S some $\mathbf{T}'$ of finite length always exists: since there is no nesting of temporal operators, each s-stable model $\mathbf{T}$ of Γ is strongly ultimately periodic, i.e., of the form $\mathbf{T} = \mathbf{T}^f \cdot X^\omega$, where $\mathbf{T}^f$ is a finite prefix of $\mathbf{T}$ of length k and X^ω an infinite repetition of a set $X \subseteq \mathcal{A}$ (Bozzelli and Pearce 2015). If we assume without loss of generality that $\mathbf{T}^f$ has length at least 2 and ends with X, then we obtain that $\mathbf{T}^f$ is a finite s-stable model of Γ as well. Indeed, as there is no temporal nesting, for each temporal subformula ψ occurring in Γ, we have that $\mathbf{T}, 0 \models \psi$ iff $\mathbf{T}^f, 0 \models \psi$: satisfaction of subformulas of the form $\circ\psi'$ depends only on position $i = 1$, and by the infinite repetition of the last state of $\mathbf{T}^f$, satisfaction of temporal subformulas $\varphi_1 \,\mathbf{U}\, \varphi_2$, $\varphi_1 \,\mathbf{R}\, \varphi_2$ at position $i = 1$ remains invariant. Hence, $\mathbf{T} \models \Gamma$ iff $\mathbf{T}^f \models \Gamma$. Furthermore, any $\langle \mathbf{H}^f, \mathbf{T}^f\rangle$, where $\mathbf{T}^f \downarrow^{id} \mathbf{H}^f$ such that $\langle \mathbf{H}^f, \mathbf{T}^f\rangle \models \Gamma$ can be extended to $\langle \mathbf{H}, \mathbf{T}\rangle$ such that $\langle \mathbf{H}, \mathbf{T}\rangle \models \Gamma$, where $\mathbf{H} = \mathbf{H}^f \cdot Y^\omega$ and $\mathbf{H}^f$ ends with Y; as $\mathbf{T}$ is s-stable, it follows $\mathbf{H}^f = \mathbf{T}^f$ and thus $\mathbf{T}^f$ is s-stable. □ □

We remark that in the argument for this proposition, the finiteness of $\mathcal{A}$ is essential.

Example 17 *The theory $\Gamma = \{\neg\neg p_i \rightarrow p_i,\ p_i \rightarrow p_{i+1} \mid i \geq 1\}$ has infinitely many (atemporal) stable models $S_i = \{p_j \mid j \geq i\}$, which form a decreasing chain $S_1 \supset S_2 \supset S_3 \cdots$; hence Γ has corresponding s-stable models (in fact, of the form $S_i \cdot \emptyset^*$ resp. $S_i \cdot \emptyset^\omega$), none of which is c-stable. However, Γ has a further atemporal stable model, viz. $\emptyset$, which is also an s-stable model of Γ and thus trivially c-stable. If we add $\neg q \rightarrow \bot$ and $p_i \rightarrow q$, for all $i \geq 1$, to Γ, the s-stable models of the resulting theory Γ' are given by $S_i \cup \{q\} \cdot (\emptyset^* + \emptyset^\omega)$, and no c-stable model exists.*

When we exclude the $\circ$-operator from THT_1, then we obtain from Propositions 26 and 29 that the c-stable models amount to the a-stable models with minimal positive information.

Proposition 28 *For a theory Γ consisting of $\circ$-free THT_1 formulas over a finite alphabet, the c-stable models coincide with the $\prec$-minimal a-stable models.*

Proof. Assume first that $\mathbf{T}$ is a c-stable model of Γ. By Proposition 26, $\mathbf{T}$ is a-stable. Towards a contradiction, assume $\mathbf{T}$ is not $\prec$-minimal among the a-stable models, i.e., some a-stable model $\mathbf{T}' \prec \mathbf{T}$ exists. By Theorem 2, we know that $\mathbf{T}'$ is also s-stable, then by Proposition 29, some c-stable model $\mathbf{T}'' \preceq \mathbf{T}'$ exists; as $\mathbf{T}'' \prec \mathbf{T}$, this contradicts that $\mathbf{T}$ is a c-stable model of Γ.

Conversely, assume $\mathbf{T}$ is a $\prec$-minimal a-stable model of Γ. By Proposition 29, some c-stable model $\mathbf{T}' \preceq \mathbf{T}$ exists, and by Proposition 26 $\mathbf{T}'$ is a-stable; hence $\mathbf{T}' = \mathbf{T}$ follows, i.e., $\mathbf{T}$ is c-stable. □ □

That is, making s-stable models of finite $\circ$-free THT_1 theories syntactically compact will also yield semantic summarizations in logic-based terms.

Question:

How to define preference such that the preferred models approximate the a-stable models and would let more a-stable models of $\circ$-free theories "survive"?

Proposition 29 *For every s-stable model $\mathbf{T}$ of a theory Γ of $\circ$-free THT_1 formulas over a finite alphabet $\mathcal{A}$, there exists some c-stable model $\mathbf{T}' \preceq \mathbf{T}$ of Γ.*

Proof.(Sketch) We determine a trace $\mathbf{T}'$ with the desired property as follows. Let μ^* be the lexicographic minimal mapping μ such that $\mathbf{T} \downarrow^{\mu} \mathbf{T}'$ holds for some s-stable model $\mathbf{T}'$, i.e., the sequence $\mu(0), \mu(1), \ldots, \mu(|\mathbf{T}|-1)$ if $\mathbf{T}$ is finite resp. $\mu(0), \mu(1), \ldots$ if $\mathbf{T}$ is infinite is smallest (no μ' exists such that $\mu'(j) < \mu^*(j)$ and $\mu'(i) = \mu^*(i)$ for all $i \in [0, i]$ for some $i \geq 0$).

Let S be the set of all s-stable models $\mathbf{T}'$ such that $\mathbf{T} \downarrow^{\mu^*} \mathbf{T}'$ holds; note that S is non-empty.

We claim that S has a chain $\mathbf{T}_0 = \mathbf{T} \succ \mathbf{T}_1 \succ \cdots \mathbf{T}_m$ that can not be extended. To show this, since there is no nesting of temporal operators, for each s-stable trace $\mathbf{T}'$ of Γ there exists some position i such that each T'_j, $j \geq i$, is a stable model of the same set $\Gamma_{\mathbf{T}'}$ of atemporal formulas, which are obtained from Γ and $\mathbf{T}'$ depending on which temporal subformulas $Op\varphi$ in Γ are true respectively false at position 0 of $\mathbf{T}'$ (positions before i are used to satisfy / falsify the temporal subformulas which need some witness position in $\mathbf{T}'$); this gives us a "type" $\tau_{\mathbf{T}'}$ of the trace. Importantly, by replacing T'_j with any stable model of $\Gamma_{\mathbf{T}'}$, we always obtain another s-stable trace of $\mathbf{T}'$ of the same type. Thus, without loss of generality, we may assume that each T'_j, $j \geq i$, is a $\subseteq$-minimal stable model of $\Gamma_{\mathbf{T}'}$ (which by finiteness of $\mathcal{A}$ exists). At the positions $j' < i$, we have stable models $T'_{j'}$ of atemporal formulas $\Gamma^{j'}_{\mathbf{T}'} \supseteq \Gamma_{\mathbf{T}'}$; no such $T'_{j'}$ can thus be a strict subset of some T'_j, $j \geq i$.

(As a lemma, if T and T' are stable models of sets Γ and Γ' of atemporal formulas, respectively, such that $T \subseteq T'$ and $\Gamma \supseteq \Gamma'$, then $T = T'$. This holds since each stable model T of a theory Γ can be reconstructed by monotonic HT-inference from Γ augmented with $\neg a$ for all $a \notin \Gamma$.)

Now pick any $\mathbf{T}'$ in S of the described form. Then, every trace $\mathbf{T}'' \prec \mathbf{T}'$ in S that is of the same type ($\tau_{\mathbf{T}'}$) must be equal to $\mathbf{T}'$ on positions $j \geq i$ and strictly smaller on at least one position $j' < i$. Continuing with traces of the type $\tau_{\mathbf{T}'}$, after finitely many steps we arrive at a minimal trace $\mathbf{T}_{\tau(\mathbf{T}')}$ among the traces of type $\tau_{\mathbf{T}'}$. If some trace $\mathbf{T}'' \prec \mathbf{T}_{\tau(\mathbf{T}')}$ exists, then we continue the chain with traces of type $\tau_{\mathbf{T}''}$. Since the number of non-equivalent types is finite (as the alphabet $\mathcal{A}$ is finite), after finitely many rounds we arrive at a trace $\mathbf{T}''$ such that the chain $\mathbf{T}_0 = \mathbf{T} \succ \mathbf{T}_1 \succ \cdots \mathbf{T}_m = \mathbf{T}''$ can not be extended.

We claim that $\mathbf{T}' = \mathbf{T}_m$ is a c-stable model of Γ; it remains to show that no s-stable model $\mathbf{T}''$ of Γ exists such that $\mathbf{T}'' \prec \mathbf{T}'$. As $\mathbf{T}'' \preceq \mathbf{T}$ by transitivity of $\preceq$ and $\mathbf{T} \downarrow^{\mu^*} \mathbf{T}'$, it follows $\mathbf{T} \downarrow^{\mu} \mathbf{T}'$ for some μ that is smaller or equal to μ^*; by minimality of μ^*, it follows $\mu = \mu^*$ and that $\mathbf{T}' \downarrow^{id} \mathbf{T}''$ must hold.

As $\mathbf{T}' \neq \mathbf{T}''$ and $\mathbf{T}'' \preceq \mathbf{T}'$, it follows that $\mathbf{T}''$ is lexicographically smaller than $\mathbf{T}'$. This, however, is a contradiction to the selection of $\mathbf{T}'$. □

When we admit $\circ$, Proposition 29 extends to finite theories.

Proposition 30 *For every s-stable model $\mathbf{T}$ of a finite theory Γ of THT_1 formulas, there exists some c-stable model $\mathbf{T}' \preceq \mathbf{T}$ of Γ.*

Proof.(Sketch) Suppose without loss of generality that Γ consists of a single formula φ. For ever s-stable models of φ, only a prefix of at most length $|\varphi|$ is relevant for evaluating the $\circ$-subformulas and can impact the evaluation in the remainder of the model. Consequently, we can divide the s-stable models $\mathbf{T}' \leq \mathbf{T}$ into finitely many subsets $\mathbf{T}_\pi$, where π is a prefix of an s-stable model of length at most $|\varphi|$. Following the argument in Proposition 29, there exists some minimal element $\mathbf{T}_\pi$ for each such prefix π, and in the finite set $\{\mathbf{T}_\pi\}$ of all such prefixes, some minimal element $\mathbf{T}'$ must exist. This $\mathbf{T}'$ then satisfies $\mathbf{T}' \leq \mathbf{T}''$ for all $\mathbf{T}'' \leq \mathbf{T}$, i.e., is minimal among all $\mathbf{T}'' \leq \mathbf{T}$. □

11 Complexity

We are now ready to state the following Proposition:

Proposition 31 *Suppose Γ is a set of formulas from THT^1. Then a finite trace $\mathbf{T}$ is a c-stable model of Γ if $\mathbf{T}$ is the shortest minimal model of Γ.*

Proof. Note that the finiteness requirement is used to ensure that there exists a shortest trace. Assume $\mathbf{T}$ is the shortest minimal model of Γ. Towards a contradiction, suppose $\mathbf{T}$ is not c-stable. Then some $\mathbf{T}' \neq \mathbf{T}$ and μ exist such that $\mathbf{T} \downarrow^{\mu} \mathbf{T}'$ and $\langle \mathbf{T}', \mathbf{T}, \mu \rangle \models \Gamma$. Let i be the first position where a contraction occurs, namely $|\mu^{-}(i)| > 1$. Since all the steps before position i are integral, we can repeatedly apply the unfolding rules (b) and (c) of Lemma 11.

The idea is to unfold the formula up to position i, for instance, if the theory was $\alpha \,\mathbf{R}\, (\beta \,\mathbf{U}\, \gamma)$ and $i = 1$, we would have the theory $(\gamma \vee (\beta \wedge \circ(\beta\, \mathbf{U}\, \gamma))) \wedge (\alpha \vee \circ\alpha\, \mathbf{R}\, (\beta\, \mathbf{U}\, \gamma))$. More in general, we will have a theory that consists of a residual theory $\circ^i\Gamma^i$ and an unfolded one called $\Gamma^{0,\ldots,i-1}$.

At $i-1$ we have: $\mathbf{M}, i-1 \models \circ\Gamma^i$; so by Persistence we have that $\mathbf{T}, \mu^-(i-1) \models \circ\Gamma^i$ (where $\mu^-(i-1) = \{i-1\}$); and since $\mathbf{T}, \mu^-(i) \models \Gamma^i$, we can leave out all from $\mu^-(i)$ except the last element, and so the sub-trace $\mathbf{T}' =_{\mathrm{def}} \mathbf{T}[0,i) \cdot \mathbf{T}[\max \mu^-(i), \lambda)$ is such that $\mathbf{T}', i-1 \models \circ\Gamma^i$, and thus $\mathbf{T}', 0 \models \Gamma$.

We can conclude that $\mathbf{T}$ is c-stable because it is not possible to properly contract any segment of the T-trace into an $\mathbf{H}$ given our hypothesis of $\mathbf{T}$ being one of the shorted models of Γ. The only other way to contract it is via the id mapping. However, it is not possible. We can see this latter point by observing that for THT^1 theories minimal models coincide with stable models. □

11.1 Fragments

We consider now the fragments from the sections above.

THT^1: Already for $\circ$-free theories, we have EXPSPACE-hardness of c-stable model existence via fragments.

GP: it is open whether EXPSPACE-hardness holds. For $\circ$-free theories, this may be derived by a reduction from $\circ$-free THT^1; more specifically, in the proof of Bozzelli and Pearce, the construction is near GP and the argument aims at the existence of a minimal model (as there a minimal model exists iff a stable model exists). The crux is whether a formula $\bigvee_i a_i \wedge (b_i \to c)$ could be encoded in GP, as positive (sub)formulas ψ can be named with atoms p_ψ, without affecting the existence of a minimal model.

THT_1: Deciding c-stable model existence is NEXPTIME-c, as by Proposition 29, this is equivalent to deciding stable model existence. Notably, by (Bozzelli and Pearce 2015, Theorem IV.3) every LTL-satisfiable THT_1 formula admits a minimal LTL model, which however may not be stable.

Deciding c-stable model existence is in $\mathrm{NEXPTIME}^{\mathrm{NP}}$, as a guess for an c-stable model can be checked with an NP oracle in polynomial time. The problem is at least Σ_2^p-hard.

Deciding compact-stable model existence is easier.

Proposition 32 *Deciding whether a given* THT_1^1 *theory* Γ *has some compact-stable model is NP-c.*

Proof. Each LTL-satisfiable THT_1^1 theory admits some minimal model, which is also a s-stable model; thus by Proposition 29, Γ has some c-stable model iff Γ is LTL-satisfiable. Deciding the latter is for THT_1^1 in NP (Demri and Schnoebelen 2002). □
□

$\mathrm{THT}_1(\circ, \Diamond)$, $\mathrm{THT}(\circ, \Box)$: Deciding existence of an c-stable resp. stable model is Σ_2^p-c.

Indeed, every stable model of a theory in $\mathrm{THT}_1(\circ, \Diamond)$ and $\mathrm{THT}(\circ, \Box)$ has only few (polynomially many) nonempty positions, which by cutting them down can be made into an almost-empty stable model where all non-empty positions are in a polynomial size prefix (Bozzelli and Pearce 2015). Hence, c-stable model existence and compact-stable model existence are in Σ_2^p. The Σ_2^p-hardness is inherited by the Σ_2^p-cness of stable semantics.

Open issues: GP complexity, $\mathrm{NEXPTIME}^{\mathrm{NP}}$ improvement, THT_1^1 c-stability, compact-stability in general?

12 Related Work and Conclusion

- One of the logic used when dealing with LTL for asynchronous systems is Temporal Logic of Actions (TLA) (Lamport 1994), which is a logic designed to express in the same formula both specifications and the system. TLA has a temporal operator called action, that resembles the next operator, but it coincides with an execution of an atomic instruction and is parameterized by the variables that are allowed to stutter. For instance, a global temporal property may refer to a clock over minutes m and hours h, while a local component might have a clock over seconds s minutes m, and hours h. Allowing stuttering of m, and h in the specification makes it possible to specify both the specification and the description of the system in the same formula.

- Asynch. extensions of Hyper LTL ((Bozzelli, Peron, and Sánchez 2021)). They resort to a notion of stuttering.

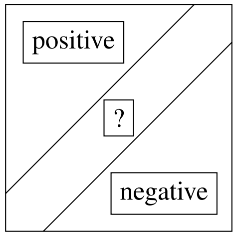

Figure 6: Possible splitting of a classical model M of a program π.

12.1 Forks

In this section, I put some observations on the semantics for both the atemporal and the temporal case where the negation is used.

12.2 Static case

Possible view: We can see the different t_i as different guesses (or views where instead of the $\mathbf{K}$ operator, we can access the different classical candidates via the negation) while minimizing h. Let us consider the program

$$\pi = \{\neg p \rightarrow q\}$$

with $T = \{t_1, t_2\}$, $t_1 = \{p, q\}$, $t_2 = \{q\}$, and $h = \emptyset$. We notice that $\neg p$ must be true in both the two worlds t_i for $i = 1, 2$ according to Defn. 7. Therefore, even if $t_2 \in T$, even in a stratified setting, we can have $h = \emptyset$. However, if we have a disjunctive (with negation) program π such that $\pi \models p$ (positive consequence) and $\pi \models \neg q$ (negative consequence), then we can evaluate the negation over p and q since all the worlds in T must be labeled by classical models of π obtaining that we can minimize h as in HT. Can a possible reading be the following one: in cTHT the implication is more *constructive* than in HT, because you may have to consider different assumptions that may have different interpretations of rule bodies?

If we add the excluded middle for p: $(p \vee \neg p)$, then p can be either always true or false for all possible set of assumptions T. Therefore, we could minimize h as it was under HT semantics.

Otherwise, if we consider T as a set of only minimal models, then if the program is stratified, we get in h what we would get in HT, and if the program has even cycles, then we may not be able to *decide* between two choices. For instance in

$$\pi' = \{\neg q \rightarrow p\} \cup \pi$$

We can have $T = \{t_2, t_3\}$ with $t_3 = \{p\}$ and empty h. While in the absence of even cycles programs like

$$\pi_2 = \{\neg p \rightarrow q \vee r\}$$

with $T = \{t_2, t_4\}$ with $t_4 = \{r\}$ do not have any h such that $\langle T, h\rangle$ is a cTHT model.

How to read negation:

- $\neg\neg p$ means p in all the accessible worlds in T
- $\neg p$ means p in any of the accessible worlds in T

13 Temporal case

13.1 T-Stuttering axioms

If $\mathbf{T} = [\emptyset, \{p\}]$, $\mathbf{H} = [\emptyset]$ and

$$\pi = \neg p \rightarrow \circ p$$

then $\langle \mathbf{T}, \mathbf{H}\rangle \models \pi$.

However, let us consider the following axiom:

$$\text{T\_stuttering: } \Box(\neg\neg p \vee \neg p)\ \forall\, p \in \mathcal{A}$$

If we add *T_stuttering:* to a theory Γ, then when minimizing, we can contract only sequences of equal states. Under this axiom, we would obtain $\mathbf{T} = [\emptyset, \{p\}]$ as c-TEL models of π.

$$\text{(A): } \Box(p \vee \neg p \vee \widehat{\circ}\top)\ \forall\, p \in \mathcal{A}$$

When applying the minimization criteria, under (A) we have that either you are contracting a total fragment, or you are locally minimized in an TEL style. Under (A) we are separating the two orthogonal minimizations, namely (i) the trace contraction, and (ii) the HT-minimization. We get therefore the stutter-free TEL models. Adding also the excluded middle, then we get stutter-free LTL models.

Axiom:

$$\Box(\neg\alpha \rightarrow \neg\alpha\ \mathbf{W}\ \widehat{\circ}\top)$$

It is a sort of $\mathbf{N}$-necessity axiom designed for negative assumptions that contribute to deriving synchronization steps. If this holds, then we have synchronization steps as expected. So, under ”reflexive” properties for negative assumptions, we gain a stutter-free TEL-like behavior.

13.2 Stratified negation for temporal programs

The temporal programs here considered are:

- fulfillment-free
- may contain constraints such as $\varphi \rightarrow \bot$
- rules have at most one atom in the head

Example 8 contains an c-unstable theory $\Gamma = \{\neg p \rightarrow \Diamond p\}$, but also $\Gamma' = \{\neg p \rightarrow \circ p\}$ is c-unstable. In this subsection, we want to address this issue, by resorting to a dependency graph. Let us first observe that the theories $\Gamma_1 = \{\neg p \rightarrow \Diamond q\}$ and $\Gamma_1' = \{\neg p \rightarrow \circ q\}$ have both c-stables models. $\mathbf{T}_2 = \emptyset \cdot \{q\}$ and $\mathbf{T}_3 = \emptyset \cdot \{q\} \cdot \emptyset$ are the c-stable models of Γ_1'. $\mathbf{T}_0 = \{q\}$, $\mathbf{T}_1 = \{q\} \cdot \emptyset$, $\mathbf{T}_2$, $\mathbf{T}_3$ are the c-stable models of Γ_1.

We note that there is a common pattern in the reason why Γ and Γ' are c-unstable, namely, that we are assuming $\neg p$ to *justify* the extension of the trace with a state where p holds. This is very similar to what happens with the logic program $\neg p \rightarrow p$, which has no answer set.

Therefore, we need a stronger notion of stratification. Let us introduce the concept of *temporal fork stratification*, under the idea of imposing the constraint that “I cannot prove that there is a next state with p assuming not p now.”

More formally, we introduce the following notion of dependency graph, where we do not only consider atoms but also synchronization steps.

Definition 16 (Dependency Graph) *The* dependency graph *of a temporal unfolded program π^ω is the directed graph $DG_\pi = \langle V, E\rangle$ where $V = \{\top^i, p^i, \neg p^i \mid p \in \mathcal{A}$ and (i) $(a,b)^{(-)} \in E$ if $a \in H(r)$ and $b \in B^+(r)$ ($b \in B^-(r)$) for some rule $\circ^k r$ in π^ω, (ii) $(q^k, p^{k'}) \in E$ for $k' \geq k$ if $\circ^k(\Box p \rightarrow q) \in \pi^\omega$, and (iii) $(q^{k'}, p^k) \in E$ for $k' \geq k$ if $\circ^k(p \rightarrow \Diamond q) \in \pi$ (iv) for each p^i add $(p^i, \top^i)$ for each $i \geq 1$.*

Intuitively, (ii) and (iii) are added because we can read $\Box p$ and $\Diamond p$ as a conjunction resp. disjunction of $\circ^i p, i \geq 0$. Condition (iv) is meant to consider the dependencies of synchronization steps. If there is a path from node n^i to node n^j such that at least one of the arcs is negative, we say that n^i *negatively depends on* n^j. If there is a path from n^i to itself such that at least one of the arcs is negative, we say that there is *a negative loop on* n^i.

Definition 17 (Temporal fork stratified program) *For a program π, we require (i) the absence of negative loops on atoms in V, (ii) for every $n^i, n^j \in V$ such that $j < i$, n^i cannot depend negatively on n^j, (iii) for every rule to be normal, and (iv) π to be fulfillment-free. If a π fulfills (i-iv), we call it a temporal fork-stratified program.*

Proposition 33 (Temporal fork stratified characterization) *Given a temporal fork stratified temporal program π, π is satisfiable under* TEL *semantics iff it is satisfiable under c*-TEL *semantics. Furthermore, if π is next-free, $\mathbf{M}$ is an c-stable model of π iff $\mathbf{M}$ is a stutter-free stable model of π.*

Given a Polarity-consistent-stratified temporal program π, since it is a temporal stratified, then its minimal LTL models are TEL models.

Since they are stratified, we can assume that there is an order in the application of an immediate consequence operator. Therefore, we can reason over a linear order and apply an inductive argument.

Let us consider a generic position i, and let us assume that synchronization steps have been proved till position $i-1$. Then, we compute the (unique) local consequence, which should be in T_i for persistence. If we have to evaluate a rule $r = B(r) \rightarrow H(r)$ with $\circ p \in H$, then if $q \in B^-(r)$, then we know that $p \neq q$ because of condition (i). More generally, $\neg q$ is not affected by what it derives for condition (i), therefore the evaluation of $\neg q$ is not compromised by any future q lately derived.

Since the initial condition does not require an initial synchronization, therefore, we can apply the same (inductive) argument for the initial case.

Counterexamples for not requiring condition (ii) on negative dependencies are Γ and Γ'.

[Davide: Can we extend this results to THT^2 *theories?]*

13.3 Unfolding of the until operator

Normal form of $\gamma \equiv \varphi \,\mathbf{U}\, \psi$ for TEL:

$$\Box(Ł_\gamma \rightarrow Ł_\psi \vee Ł_\varphi)$$
$$\Box(Ł_\gamma \rightarrow Ł_\psi \vee \circ Ł_\gamma)$$
$$\Box(Ł_\psi \rightarrow Ł_\gamma)$$
$$\Box(Ł_\varphi \wedge \circ Ł_\gamma \rightarrow Ł_\gamma)$$
$$\Box(Ł_\gamma \rightarrow \Diamond Ł_\psi)$$

We can find a way not to force any synchronicity using aux atoms:

$$\Box(\neg Ł_s \rightarrow Ł_{s'})$$
$$\Box(\neg Ł_{s'} \rightarrow Ł_s)$$
$$\Box(\neg Ł_{s'} \rightarrow \circ Ł_{s'})$$
$$\Box(\neg Ł_s \rightarrow \circ Ł_s)$$

We duplicate rules, putting $\neg Ł_s$ (and $\neg Ł_{s'}$) in the body of rules which have the next operator in the head. We obtain:

$$\Box(Ł_\gamma \rightarrow Ł_\psi \vee Ł_\varphi)$$
$$\Box(Ł_\gamma \wedge \neg Ł_s \rightarrow Ł_\psi \vee \circ Ł_\gamma)$$
$$\Box(Ł_\gamma \wedge \neg Ł_{s'} \rightarrow Ł_\psi \vee \circ Ł_\gamma)$$
$$\Box(Ł_\psi \rightarrow Ł_\gamma)$$
$$\Box(Ł_\varphi \wedge \circ Ł_\gamma \rightarrow Ł_\gamma)$$
$$\Box(Ł_\gamma \rightarrow \Diamond Ł_\psi)$$

13.4 What happens with unstratified negation?

To consider:

$$\pi = \Box(\neg p \rightarrow q) \wedge \Box(\neg q \rightarrow p)$$

in relation with

$$\pi' = \Box(p \vee q)$$

From π it looks like we cannot extend a trace based on an alternation based on the negation, i.e., such an alternation cannot be used to derive a synchronization step.

From π' we realize that we can fully positively derive either p or q without any other assumption, therefore we can alternate. Note that positive derived alternation is a justification for a trace extension.

Therefore, we could use again the dependency graph for this purpose. Namely, if alternation among a set of atoms U is positively guaranteed, then it may be possible that we can use the negation in a more unrestricted way. For instance, if π' belongs to the theory Γ, then we can use the negation over p to derive $\circ q$.

13.5 EX

To investigate

$$(\neg\neg\varphi) \wedge \psi$$

13.6 Possible applications

- Planning, where a planning domain P is encoded as a temporal program $\pi(P)$, but inertia rules r have $\circ\top$ in the body, namely $\circ\top \in B^+(r)$. Intuitively, repetitions of actions in a plan may be considered fine, but stretching the length of a plan because of inertia rules should not be possible. Inertial rules should not force any extension of the plan.

Normal form for THT_1^1 theories Let us consider a THT_1^1 theory. We can always rewrite it into a normal form via a Tsein transformation tr obtaining only formulas of the following types:

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k \to \psi_1 \vee \cdots \vee \psi_i \tag{27}$$
$$\Box(p) \leftrightarrow p_\Box \tag{28}$$
$$\Diamond(p) \leftrightarrow p_\Diamond \tag{29}$$
$$\circ(p) \leftrightarrow p_\circ \tag{30}$$

Where $\varphi_1, \ldots, \varphi_k$ and $\psi_1 \vee \cdots \vee \psi_i$ are conjunction and disjunction of atoms, either originally present in the alphabet, or later introduced by the Tsein translation.

Now we want to rewrite the above formulas into a new normal form, but this time under stable semantics preserving equi-satisfiability. In order to do that, we rewrite (30) into

$$\circ(p) \leftrightarrow (p_\circ \wedge m) \tag{31}$$

And we add also $\{\Diamond\neg p \wedge p \to m, \Diamond p \wedge \neg p \to m\}$ for every $p \in \mathcal{A}$ to the theory. We call this second transformation tr'

Claim: There exists a stable model for $tr'(\varphi)$ iff there exists a c-stable model for $tr(\varphi)$.

Proof. The left to right direction is obtained by noting that if there is a c-stable model for φ, then there is a c-stable model for $tr(\varphi)$. Furthermore, if there is a c-stable model for $tr(\varphi)$, then there is a stable model of $tr(\varphi)$. Therefore, also one for

The right to left. If you derive m, then you derive a non-contraction step. Let us assume for the sake of contradiction that $\mathbf{T}$ is one of the shortest stable models for $tr'(\varphi)$ and that $m \in T_0$, but T_0 can be contracted, namely that there exist $\mathbf{H}$ and μ such that $|\mu^-(0)| > 1$ such that $\langle \mathbf{H}, \mathbf{T}, \mu\rangle \models \varphi$.

Let us consider all three ways in which we can derive m and the conclude a contradiction for all of them. □

14 Stutter minimality

- Define a less-stutter ordering
- Define stutter-minimal temporal equilibrium models
- Under which conditions, contracting TEMs are the stutter-minimal TEMs and vice versa
- find an example of contracting TEM that is not stutter-minimal (using $\circ$ and without using $\circ$)
- Study stutter-minimality under the Excluded Middle axiom for all atoms. Which is the relation to stutter-invariance in LTL?

15 Metric

Hint for metric: I suspect that the time stamp function τ_w should not assign $\tau_h(i) = \tau_t(\mu(i))$ but $\tau_h(i) = \tau_t(\mu(i+1)-1)$ instead (if $i+1$ is a valid position), that is the last $\mathbf{T}$-state "covered" by H_i. In the example, this means for instance that $\tau_h(0) = \tau_t(1)$ and $\tau_h(1) = \tau_t(4)$. When $\mathbf{T}$ is infinite but $\mathbf{H}$ is not, the final state of the latter H_i with $i = \lambda_h - 1$ would have $\tau_h(\lambda_h - 1) = \tau_t(\mu(\lambda_h) - 1) = \tau_t(\omega - 1)$ but $\omega - 1$ is not defined. In that case, we could make $\tau_t(i)$ either undefined or perhaps ω meaning that this last state is an abbreviation of the limit of the infinite sequence of remaining states in $\mathbf{T}$, and so, its time stamp is also a limit ordinal. So, in the example, $\tau_h(2) = \omega$.

TE: I am not sure about this. How is then $\diamond\varphi$ handled (one misses the states before $\tau_t(\mu(i+1)-1$? Note that the next operator $\circ\varphi$ is, for metric temporal logic, a particular case of the interval "cross-diamond"$_{1,1]}\varphi$ if I am not mistaken. So in general one would need to think about how intervals in here are mapped to intervals in there (?).

16 Examples

Example 18 (Not correct) *Consider the traces $\mathbf{H} = \emptyset \cdot \emptyset \cdot \{a\}^\omega$ and $\mathbf{T} = \emptyset \cdot \{a\}^\omega$. We have $\mathbf{H} \preceq \mathbf{T}$ because $\mathbf{T} \downarrow^{id} \mathbf{H}$, but also $\mathbf{T} \preceq \mathbf{H}$ because we can use $\mathbf{H} \downarrow^{\mu} \mathbf{T}$ with $\mu(0) = 0$ and $\mu(i) = i - 1$ for all $i \geq 1$. Now, consider the formula:*

$$\circ\circ a \wedge \circ\circ\Box(a \to \circ a) \tag{32}$$

Note that this theory forces an infinite trace. In fact, the only (synchronous) t-stable model of (32) *is* $\mathbf{H}$ *and it seems that this should also be the only c-stable model: there is no reason to conclude a at step $i = 1$. Yet, we have both $\langle \mathbf{H}, \mathbf{T}, id\rangle, 0 \models$ (32) and $\langle \mathbf{T}, \mathbf{H}, \mu\rangle, 0 \models$ (32). Apparently, the fact $\mathbf{T} \downarrow^{id} \mathbf{H}$ has precedence over $\mathbf{H} \downarrow^{\mu} \mathbf{T}$.* □

Example 19 *Take again the traces $\mathbf{H} = \emptyset \cdot \emptyset \cdot \{a\}^\omega$ and $\mathbf{T} = \emptyset \cdot \{a\}^\omega$. Consider now the formula:*

$$\neg a \wedge \circ(a \vee \neg a) \wedge \circ\circ a \wedge \circ\circ\Box(a \to \circ a) \tag{33}$$

The two t-stable models of this formula are $\mathbf{H}$ *and* $\mathbf{T}$. *I had made this observation before:*

> If we go for equivalence classes, both $\mathbf{H}$ and $\mathbf{T}$ would be stable. But if we use Def2 then none would be stable

but this is wrong. We cannot form a model by contracting $\mathbf{T}$ to $\mathbf{H}$ or vice versa because the formula $\circ(a \vee \neg a)$ forces us to have $H_1 = T_1$.

These examples would lead us to the following definition

Definition 18 (Temporal Equilibrium Model (A2')) *A total* cTHT*-trace $\langle \mathbf{T}, \mathbf{T}, id\rangle$ is a* temporal equilibrium model *of a theory Γ if it is a model of Γ (that is $\mathbf{T}, 0 \models \Gamma$ in* LTL*) and for all $\mathbf{H}$ and μ such that $\langle \mathbf{H}, \mathbf{T}, \mu\rangle, 0 \models \Gamma$ we have $\mathbf{H} \downarrow^{id} \mathbf{T}$.*

In the following example, we want to study a new source of instability with respect to the usual TEL semantics.

Example 20 *Modified after Meeting (18.Sept) Let us consider the following formulas $\varphi_1 = \Box(\neg\circ\top \to \circ\top)$, $\varphi_2 = \neg\circ\top \to \circ\top$, and $\varphi_3 = \Box\Diamond\circ\top$. The unique* TEL *and* LTL *models of φ_1 are traces of length $\lambda = \omega$. However, let us propose the following interpretation $\langle \mathbf{H}, \mathbf{T}, \mu\rangle$, where $\mathbf{H} = \emptyset$, and $\mathbf{T} = \emptyset^\omega$. Note that the only possible way to define μ is $\mu(i) = 0$ for each $i \in [0, \ldots \lambda)$. Applying the definition of the semantics of c*THT*, we obtain that $\langle \mathbf{H}, \mathbf{T}, \mu\rangle, 0 \models \varphi_1$. Therefore, we have found a source of instability. We could try to* stabilize *φ_1. Here follows some possible ways:*

- *adding $\Box\circ\top$, i.e., we are asserting that the trace is infinite, therefore you are justifying what classically follows from φ_1;*

- *adding a choice $\Box(p \vee \neg p)$, since the alternation works as a justification of an extension of the trace.*

φ_1 seems to require a justification for extending the trace, while φ_3 doesn't.

Example 21 *Let us consider the following formulas $\varphi_1 = \Box\Diamond\neg\circ\top$, and $\varphi_2 = \Box\Diamond\neg\neg\circ\top$. Given an interpretation $\langle \mathbf{H}, \mathbf{T}, \mu \rangle$, and a time point $i \in [0, \ldots, \lambda)$, then $\langle \mathbf{H}, \mathbf{T}, \mu \rangle, i \models \varphi_1$ implies that there exists $j \geq i$ such that $\langle \mathbf{H}, \mathbf{T}, \mu \rangle, j \models \neg\circ\top$. Note that $\langle \mathbf{H}, \mathbf{T}, \mu \rangle, j \models \neg\circ\top$ is equivalent to $\langle \mathbf{T}, \mathbf{T}, id \rangle, j' \not\models \circ\top$ for each $j' \in \lambda^{-1}(j)$, therefore we are requiring the $\mathbf{T}$ trace to be finite. Furthermore, if $\mathbf{H} = \emptyset$, then $\langle \mathbf{H}, \mathbf{T}, \mu \rangle, 0 \models \Box\Diamond\neg\circ\top$. Therefore, the unique c-stable model is $\mathbf{T} = \emptyset$.*

φ_2 requires $\mathbf{T}$ to be infinite, while $\mathbf{H}$ could be either finite or infinite. Therefore, there are no c-stable models.

`Claim`: If there are no temporal operators within the scope of negation (or dobule implication), then there exists a T-stable model iff there exists a stable model.