Logic-based order-of-magnitude qualitative reasoning for closeness via proximity intervals

Alfredo Burrieza   Emilio Muñoz-Velasco   Manuel Ojeda-Aciego

Universidad de Málaga. Andalucía Tech

Feb 23, 2015

Qualitative Reasoning (QR)

- QR is very useful for searching solutions to problems about the behavior of physical systems without using differential equations or exact numerical data.
- It is possible to reason about incomplete knowledge by providing an abstraction of the numerical values.
- QR has applications in AI, such as Robot Kinematics, Data Analysis, and dealing with movements.

Logics and QR

- First papers have been focused on
  - Spatio-Temporal Reasoning and
  - about solutions of ordinary differential equations
- Our work has been focused on Order of Magnitude QR.

Order of Magnitude QR

- A partition of the real line in qualitative classes (small, medium, large,. . .) is considered. The absolute approach.
- A family of binary order of magnitude relations which establishes different comparison relations (negligibility, closeness, comparability, . . .). The relative approach.
- We defined some logics which bridge the absolute and relative approaches.

Previous works I

- Sound and complete multimodal logics dealing with negligibility, comparability, non-closeness and distance.
  - A multimodal logic approach to order of magnitude qualitative reasoning with comparability and negligibility relations. *Fundamenta Informaticae*, 68:21–46, 2005.
  - A Logic for Order of Magnitude Reasoning with Negligibility, Non-closeness and Distance. *Lecture Notes in Computer Science* 4788: 210–219, 2007
- Theorem provers for logics dealing with negligibility, non-closeness and distance.
  - (with A. Mora, and E. Orłowska) An implementation of a dual tableaux system for order-of-magnitude qualitative reasoning. *Intl J on Computer Mathematics* 86:1852–1866, 2009
  - (with J Golinska) Relational approach for a logic for order of magnitude qualitative reasoning with negligibility, non-closeness and distance. *Logic Journal of the IGPL* 17(4): 375–394, 2009

Previous works II

- (with J Golinska) Dual tableau for a multimodal logic for order of magnitude qualitative reasoning with bidirectional negligibility. *Intl J on Computer Mathematics* 86: 1707–1718, 2009

- Sound, complete and decidable PDL for qualitative velocity, and for dealing with movements.
  - A logic framework for reasoning with movement based on fuzzy qualitative representation. *Fuzzy Sets and Systems*, 242:114–131, 2014.
  - (with J Golinska) Reasoning with Qualitative Velocity: Towards a Hybrid Approach. *Lecture Notes in Computer Science* 7208: 635–646 2011
  - A PDL approach for qualitative velocity. *Intl J of Uncertainty, Fuzziness, and Knowledge-based Systems*, 19(1):11–26, 2011
  - Closeness and distance in order of magnitude qualitative reasoning via PDL. *Lecture Notes in Artificial Intelligence* 5988:71–80, 2010.
- We focus here on a multimodal logic for **closeness**.

Why this approach?

- So far, the only published reference on a logic-based approach to closeness uses PDL and qualitative sum.
  Specifically, two values are assumed to be close if one of them can be obtained from the other by adding a “small” number, and small numbers are defined as those belonging to a fixed interval.
- This specific approach has a number of potential applications but might not be so useful in other situations, for instance, when there are barriers (physical, temporal, etc.).
- In this work, we consider a new logic-based alternative to the notion of closeness in the context of multimodal logics. Our notion of closeness stems from the idea that two values are considered to be *close* if they are inside a prescribed area or *proximity interval*.
- This leads to an equivalence relation, particularly, transitivity holds.

Preliminary definitions

We will consider a strictly ordered set of real numbers $(\mathbb{S}, <)$ divided into the following qualitative classes:

$$\text{NL} = (-\infty, -\gamma) \qquad \text{PS} = (+\alpha, +\beta]$$
$$\text{NM} = [-\gamma, -\beta) \qquad \text{INF} = [-\alpha, +\alpha] \qquad \text{PM} = (+\beta, +\gamma]$$
$$\text{NS} = [-\beta, -\alpha) \qquad \text{PL} = (+\gamma, +\infty)$$

Note that all the intervals are considered relative to $\mathbb{S}$.
We will consider each qualitative class to be divided into disjoint intervals called *proximity intervals*, as shown in the figure below. The qualitative class INF is itself a proximity interval.

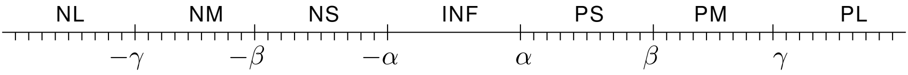

Figure : Proximity intervals.

Preliminary definitions

Definition

Let $(\mathbb{S}, <)$ be a strictly linear divided into the qualitative class defined above.

- A *proximity structure* is a finite set $\mathcal{I}(\mathbb{S}) = \{I_1, I_2, \ldots, I_n\}$ of intervals in $\mathbb{S}$, such that:
  1. For all $I_i, I_j \in \mathcal{I}(\mathbb{S})$, if $i \neq j$, then $I_i \cap I_j = \varnothing$.
  2. $I_1 \cup I_2 \cup \cdots \cup I_n = \mathbb{S}$.
  3. For all $x, y \in \mathbb{S}$ and $I_i \in \mathcal{I}(\mathbb{S})$, if $x, y \in I_i$, then x, y belong to the same qualitative class.
  4. $\mathrm{INF} \in \mathcal{I}(\mathbb{S})$.
- Given a proximity structure $\mathcal{I}(\mathbb{S})$, the binary relation of closeness $\mathfrak{c}$ is defined, for all $x, y \in \mathbb{S}$, as follows: $x \,\mathfrak{c}\, y$ if and only if there exists $I_i \in \mathcal{I}(\mathbb{S})$ such that $x, y \in I_i$.

Preliminary definitions

From now on, we will denote by $\mathcal{Q} = \{\mathsf{NL}, \mathsf{NM}, \mathsf{NS}, \mathsf{INF}, \mathsf{PS}, \mathsf{PM}, \mathsf{PL}\}$ the set of qualitative classes, and by QC to any element of $\mathcal{Q}$.

Definition

Let $(\mathbb{S}, <)$ be a strictly linear set divided into the qualitative classes defined above. The binary relation of negligibility $\mathfrak{n}$ is defined on $\mathbb{S}$ as $x \,\mathfrak{n}\, y$ if and only if one of the following situations holds:

(*i*) $x \in \mathsf{INF}$ and $y \notin \mathsf{INF}$,

(*ii*) $x \in \mathsf{NS} \cup \mathsf{PS}$ and $y \in \mathsf{NL} \cup \mathsf{PL}$.

Preliminary results

Proposition

The relation $\mathfrak{c}$ defined above has the following properties:

1. $\mathfrak{c}$ *is an equivalence relation on* $\mathbb{S}$.
2. *For all* $x, y, z \in \mathbb{S}$, *the following holds:*
   (a) *If* $x, y \in \text{INF}$, *then* $x \; \mathfrak{c} \; y$.
   (b) *For every* $\text{QC} \in \mathcal{Q}$, *if* $x \in \text{QC}$ *and* $x \; \mathfrak{c} \; y$, *then* $y \in \text{QC}$.

Proposition

For all $x, y, z \in \mathbb{S}$ *we have:*

(i) *If* $x \; \mathfrak{c} \; y$ *and* $y \; \mathfrak{n} \; z$, *then* $x \; \mathfrak{n} \; z$.
(ii) *If* $x \; \mathfrak{n} \; y$ *and* $y \; \mathfrak{c} \; z$, *then* $x \; \mathfrak{n} \; z$.

The language $\mathcal{L}(MQ)^{\mathcal{P}}$

Introducing the Syntax

Modal connectives $\overrightarrow{\Box}$ and $\overleftarrow{\Box}$ to deal with the usual ordering $<$.
Two other modal operators will be used, $\boxed{c}$ for closeness, and $\boxed{n}$ for negligibility.
Their informal meanings are the following

- $\overrightarrow{\Box}A$ means *A is true in every point greater than the current one*.
- $\overleftarrow{\Box}A$ means *A is true in every point smaller than the current one*.
- $\boxed{c}A$ means *A is true in every point close to the current one*
- $\boxed{n}A$ means *A is true in every point negligible with respect to the current one*.

The language $\mathcal{L}(MQ)^{\mathcal{P}}$

Syntax

The formulas are defined as follows:

$$A = p \mid \xi \mid c_i \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid \overrightarrow{\Box} A \mid \overleftarrow{\Box} A \mid \boxed{\text{n}} A \mid \boxed{\text{c}} A$$

where

- p represents the propositional variables
- ξ is a metavariable denoting any milestone $\alpha^-, \alpha^+, \beta^-, \beta^+, \gamma^-, \gamma^+$
- c_i are proximity constants (finitely many)
- The connectives $\neg, \wedge, \vee$ and $\rightarrow$ are the classical ones
- $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\boxed{\text{n}}$, $\boxed{\text{c}}$ are the previous unary modalities

We will use the symbols $\overrightarrow{\Diamond}, \overleftarrow{\Diamond}, \textcircled{c}, \textcircled{n}$ as abbreviations. We will also introduce abbreviations for qualitative classes, for instance, ps for $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}\beta^+) \vee \beta^+$.

The language $\mathcal{L}(MQ)^{\mathcal{P}}$

Semantics

Definition

A *frame* for $\mathcal{L}(MQ)^{\mathcal{P}}$ is a tuple $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P})$, where:

1. $(\mathbb{S}, <)$ is a strict linearly ordered set.
2. $\mathcal{D} = \{+\alpha, -\alpha, +\beta, -\beta, +\gamma, -\gamma\}$ is a set of designated points in $\mathbb{S}$ (called *frame constants*).
3. $\mathcal{I}(\mathbb{S})$ is a proximity structure.
4. $\mathcal{P}$ is a bijection (called *proximity function*), $\mathcal{P} \colon \mathcal{C} \longrightarrow \mathcal{I}(\mathbb{S})$, that assigns to each proximity constant c a proximity interval.

The language $\mathcal{L}(MQ)^{\mathcal{P}}$

Semantics

Definition

Let Σ be a frame for $\mathcal{L}(MQ)^{\mathcal{P}}$, an *MQ- model* is an ordered pair $\mathcal{M} = (\Sigma, h)$, where h is a *meaning function* (or, *interpretation*) $h\colon \mathcal{V} \longrightarrow 2^{\mathbb{S}}$.
Any interpretation can be uniquely extended to the set of all formulas in $\mathcal{L}(MQ)^{\mathcal{P}}$ (also denoted by h) as follows:

$$
\begin{array}{rcl}
h(\overrightarrow{\Box} A) & = & \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } x < y\} \\
h(\overleftarrow{\Box} A) & = & \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } y < x\} \\
h(\boxed{c} A) & = & \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } x\, \mathfrak{c}\, y\} \\
h(\boxed{n} A) & = & \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } x\, \mathfrak{n}\, y\} \\
h(\alpha^+) & = & \{+\alpha\} \qquad h(\beta^+) = \{+\beta\} \qquad h(\gamma^+) = \{+\gamma\} \\
h(\alpha^-) & = & \{-\alpha\} \qquad h(\beta^-) = \{-\beta\} \qquad h(\gamma^-) = \{-\gamma\} \\
h(c_i) & = & \{x \in \mathbb{S} \mid x \in \mathcal{P}(c_i)\}
\end{array}
$$

The definitions of *truth*, *satisfiability* and *validity* are the usual ones.

An axiom system for $\mathcal{L}(MQ)^{\mathcal{P}}$

The axiom system $MQ^{\mathcal{P}}$ consists of all the tautologies of classical propositional logic plus the following axiom schemata and rules of inference:

For white connectives

K1 $\vec{\Box}(A \to B) \to (\vec{\Box}A \to \vec{\Box}B)$

K2 $A \to \vec{\Box}\overleftarrow{\Diamond}A$

K3 $\vec{\Box}A \to \vec{\Box}\vec{\Box}A$

K4 $(\vec{\Box}(A \vee B) \wedge \vec{\Box}(\vec{\Box}A \vee B) \wedge \vec{\Box}(A \vee \vec{\Box}B)) \to (\vec{\Box}A \vee \vec{\Box}B)$

For frame constants

c1 $\overleftarrow{\Diamond}\xi \vee \xi \vee \vec{\Diamond}\xi$

c2 $\xi \to (\overleftarrow{\Box}\neg\xi \wedge \vec{\Box}\neg\xi)$

c3 $\gamma^- \to \vec{\Diamond}\beta^-$

c4 $\beta^- \to \vec{\Diamond}\alpha^-$

c5 $\alpha^- \to \vec{\Diamond}\alpha^+$

c6 $\alpha^+ \to \vec{\Diamond}\beta^+$

c7 $\beta^+ \to \vec{\Diamond}\gamma^+$

An axiom system (cont'd)

For proximity constants (for all $i, j \in \{1, \ldots, r\}$)

p1 $\bigvee_{i=1}^{r} c_i$

p2 $c_i \to \neg c_j$ (for $i \neq j$)

p3 $(\overleftarrow{\Diamond} c_i \wedge \overrightarrow{\Diamond} c_i) \to c_i$

p4 $\overleftarrow{\Diamond} c_i \vee c_i \vee \overrightarrow{\Diamond} c_i$

Mixed axioms (for all $i \in \{1, \ldots, r\}$)

m1 $(c_i \wedge \mathrm{qc}) \to (\overleftarrow{\Box}(c_i \to \mathrm{qc}) \wedge \overrightarrow{\Box}(c_i \to \mathrm{qc}))$

m2 $(c_i \wedge \mathrm{inf}) \to (\overleftarrow{\Box}(\mathrm{inf} \to c_i) \wedge \overrightarrow{\Box}(\mathrm{inf} \to c_i))$

m3 $\boxed{\scriptstyle c} A \leftrightarrow \left(A \wedge \bigvee_{i=1}^{r} \left(c_i \wedge (\overleftarrow{\Box}(c_i \to A) \wedge \overrightarrow{\Box}(c_i \to A))\right)\right)$

m4 $\boxed{\scriptstyle n} A \leftrightarrow \left(\left(\mathrm{inf} \to (\overleftarrow{\Box}(\neg\mathrm{inf} \to A) \wedge \overrightarrow{\Box}(\neg\mathrm{inf} \to A))\right) \wedge \left((\mathrm{ns} \vee \mathrm{ps}) \to (\overleftarrow{\Box}(\mathrm{nl} \to A) \wedge \overrightarrow{\Box}(\mathrm{pl} \to A))\right)\right)$

An axiom system (cont'd)

The mirror images of **K1, K2** and **K4** are also considered as axioms.

Rules of inference:

(MP) Modus Ponens for $\rightarrow$.

(N$\overrightarrow{\Box}$) If $\vdash A$ then $\vdash \overrightarrow{\Box} A$.

(N$\overleftarrow{\Box}$) If $\vdash A$ then $\vdash \overleftarrow{\Box} A$.

The syntactical notions of *theoremhood* and *proof* for $MQ^{\mathcal{P}}$ are defined as usual.

Completeness

- We follow the step-by-step method, which is a Henkin-style proof.
- The idea is to show that for any consistent formula A, a model for A can be built, and this is done by successive finite approximations.
- It is worth to note that the actual construction of the successive finite approximations has a number of specific (and interesting) problems, mainly related to the need of the proximity functions within a frame.

Theorem (Completeness)

If A is valid formula of $\mathcal{L}(MQ)^{\mathcal{P}}$, then A is a theorem of $MQ^{\mathcal{P}}$.

Completeness

- We follow the step-by-step method, which is a Henkin-style proof.
- The idea is to show that for any consistent formula A, a model for A can be built, and this is done by successive finite approximations.
- It is worth to note that the actual construction of the successive finite approximations has a number of specific (and interesting) problems, mainly related to the need of the proximity functions within a frame.

Theorem (Completeness)

If A is valid formula of $\mathcal{L}(MQ)^{\mathcal{P}}$, then A is a theorem of $MQ^{\mathcal{P}}$.

Decidability

- The idea is to show that $MQ^{\mathcal{P}}$ has the strong finite model property.
- Firstly, we show the soundness and completeness of $MQ^{\mathcal{P}}$ wrt a class of models weaker than the *MQ*-models.
- *MQ*-models do not serve our purpose in order to prove the strong finite model property of $MQ^{\mathcal{P}}$ because there are formulas which are satisfiable just in infinite *MQ*-models (since *MQ*-models are strict linear orders).
- The definition of the (weaker) *MQC*-models is a generalization of that of *MQ*-models in which the irreflexivity is restricted just to the milestones.

Theorem (Strong Finite Model Property)

Let A be a formula of $\mathcal{L}(MQ)^{\mathcal{P}}$. If A^* is satisfiable in a MQC-model, then A^* is satisfiable in a finite MQC-model containing at most 2^n points, where n is the number of subformulas of A^*.

Theorem (Decidability)

$MQ^{\mathcal{P}}$ is decidable.

Decidability

- The idea is to show that $MQ^{\mathcal{P}}$ has the strong finite model property.
- Firstly, we show the soundness and completeness of $MQ^{\mathcal{P}}$ wrt a class of models weaker than the *MQ*-models.
- *MQ*-models do not serve our purpose in order to prove the strong finite model property of $MQ^{\mathcal{P}}$ because there are formulas which are satisfiable just in infinite *MQ*-models (since *MQ*-models are strict linear orders).
- The definition of the (weaker) *MQC*-models is a generalization of that of *MQ*-models in which the irreflexivity is restricted just to the milestones.

Theorem (Strong Finite Model Property)

Let A be a formula of $\mathcal{L}(MQ)^{\mathcal{P}}$. If A^* is satisfiable in a MQC-model, then A^* is satisfiable in a finite MQC-model containing at most 2^n points, where n is the number of subformulas of A^*.

Theorem (Decidability)

$MQ^{\mathcal{P}}$ is decidable.

Decidability

- The idea is to show that $MQ^{\mathcal{P}}$ has the strong finite model property.
- Firstly, we show the soundness and completeness of $MQ^{\mathcal{P}}$ wrt a class of models weaker than the MQ-models.
- MQ-models do not serve our purpose in order to prove the strong finite model property of $MQ^{\mathcal{P}}$ because there are formulas which are satisfiable just in infinite MQ-models (since MQ-models are strict linear orders).
- The definition of the (weaker) MQC-models is a generalization of that of MQ-models in which the irreflexivity is restricted just to the milestones.

Theorem (Strong Finite Model Property)

Let A be a formula of $\mathcal{L}(MQ)^{\mathcal{P}}$. If A^* is satisfiable in a MQC-model, then A^* is satisfiable in a finite MQC-model containing at most 2^n points, where n is the number of subformulas of A^*.

Theorem (Decidability)

$MQ^{\mathcal{P}}$ is decidable.

Future work

- Study the complexity of the logic.
- Develop automated provers for this logic: Rasiowa-Sikorsky ??
- Implement those provers.

Logic-based order-of-magnitude qualitative reasoning for closeness via proximity intervals

Alfredo Burrieza Emilio Muñoz-Velasco Manuel Ojeda-Aciego

Universidad de Málaga. Andalucía Tech

Feb 23, 2015

Completeness

Maximal consistency lemmas

The notions of *consistency* and *maximal consistency* for $MQ^{\mathcal{P}}$ are the usual ones; $\mathcal{MC}$ will denote the set of all mc-sets of formulas.

Definition

The relations $\rhd$ and $\sim$ are defined on $\mathcal{MC}$ as follows:

- $\Gamma_1 \rhd \Gamma_2$ if and only if $\{A \mid \overrightarrow{\Box} A \in \Gamma_1\} \subseteq \Gamma_2$.
- $\Gamma_1 \sim \Gamma_2$ if and only if $\Gamma_1 \rhd \Gamma_2$ or $\Gamma_1 = \Gamma_2$ or $\Gamma_2 \rhd \Gamma_1$.

Completeness

Maximal consistency lemmas

Lemma

1. $\Gamma_1 \rhd \Gamma_2$ *if and only if* $\{A \mid \overleftarrow{\Box} A \in \Gamma_2\} \subseteq \Gamma_1$.
2. $\Gamma_1 \rhd \Gamma_2$ *iff* $\{\overrightarrow{\Diamond} A \mid A \in \Gamma_2\} \subseteq \Gamma_1$ *iff* $\{\overleftarrow{\Diamond} A \mid A \in \Gamma_1\} \subseteq \Gamma_2$.
3. $\rhd$ *is a transitive relation on* $\mathcal{MC}$.
4. *If* $\Gamma_1 \rhd \Gamma_2$ *and* $\Gamma_1 \rhd \Gamma_3$, *then* $\Gamma_2 \sim \Gamma_3$, *for all* $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$.
5. *If* $\Gamma_2 \rhd \Gamma_1$ *and* $\Gamma_3 \rhd \Gamma_1$, *then* $\Gamma_2 \sim \Gamma_3$, *for all* $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$.

Lemma

1. *Given* $\Gamma \in \mathcal{MC}$ *there is exactly one proximity constant* $c \in \mathcal{C}$ *such that* $c \in \Gamma$.
2. *For all* $\Gamma_i \in \mathcal{MC}$ *and* $c \in \mathcal{C}$, *if* $\Gamma_1 \rhd \Gamma_2 \rhd \Gamma_3$ *and* $c \in \Gamma_1, \Gamma_3$, *then* $c \in \Gamma_2$.

Completeness

Maximal consistency lemmas

Lemma (Lindenbaum Lemma)

Any consistent set of formulas in $MQ^{\mathcal{P}}$ can be extended to an mc-set in $MQ^{\mathcal{P}}$.

Lemma

Assume $\Gamma_1 \in \mathcal{MC}$. Then:

1. *If $\overrightarrow{\Diamond} A \in \Gamma_1$, then there exists $\Gamma_2 \in \mathcal{MC}$ such that $\Gamma_1 \rhd \Gamma_2$ and $A \in \Gamma_2$.*
2. *If $\overleftarrow{\Diamond} A \in \Gamma_1$, then there exists $\Gamma_2 \in \mathcal{MC}$ such that $\Gamma_2 \rhd \Gamma_1$ and $A \in \Gamma_2$.*

Completeness

Step-by-step approach to completeness

The specific construction of the successive approximations of the required model for a consistent formula A forces us to consider the following weaker version of the notion of frame:

Definition

Given a denumerable infinite set $\mathcal{S}$, a *partial frame* is a tuple $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P})$ where $\mathbb{S}$ is a subset of $\mathcal{S}$, $\mathcal{D}$ is a set of designated points in $\mathbb{S}$, $<$ is a total strict ordering on $\mathbb{S}$, $\mathcal{I}(\mathbb{S})$ is a proximity structure, and $\mathcal{P}\colon \mathcal{C} \to \mathcal{I}(\mathbb{S})$ is a partial bijective function where $\mathcal{C}$ is the set of proximity constants.

Completeness

Conditionals

Definition

Let $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P})$ be a partial frame.

1. A *trace* of Σ is a function $f_\Sigma : \mathbb{S} \longrightarrow 2^{\mathcal{L}(MQ)^{\mathcal{P}}}$ such that for all $x \in \mathbb{S}$ the set $f_\Sigma(x)$ is a maximal consistent set.
2. A trace of Σ, f_Σ, is called:
   - *Coherent* if it satisfies for all $x, y \in \mathbb{S}$ and $\xi \in \mathcal{D}$:
     1. $\xi^+ \in f_\Sigma(+\xi)$ and $\xi^- \in f_\Sigma(-\xi)$
     2. If $x < y$, then $f_\Sigma(x) \rhd f_\Sigma(y)$
     3. Let $c_i \in \mathcal{C}$ and $I \in \mathcal{I}(\mathbb{S})$. If $c_i \in f_\Sigma(x)$ and $x \in I$, then $\mathcal{P}(c_i) = I$.
   - *Full* if it is coherent and, for all formulas A, and all $x \in \mathbb{S}$, it satisfies the following conditions:
     - **(a)** if $\overrightarrow{\Diamond} A \in f_\Sigma(x)$, there exists y such that $x < y$ and $A \in f_\Sigma(y)$
     - **(b)** if $\overleftarrow{\Diamond} A \in f_\Sigma(x)$, there exists y such that $y < x$ and $A \in f_\Sigma(y)$

The expressions (a) (resp., (b)) are called *prophetic* (resp., *historic*).
A prophetic conditional is said to be *active* if $\overrightarrow{\Diamond} A \in f_\Sigma(x)$, but there is no y such that $x < y$ and $A \in f_\Sigma(y)$; otherwise, is said to be *exhausted*.

Completeness

Main results

Lemma (Trace lemma)

Let f_Σ be a full trace of a frame Σ. Let h be an interpretation assigning to each propositional variable p the set $h(p) = \{x \in \mathbb{S} \mid p \in f_\Sigma(x)\}$. Then, for any formula A we have $h(A) = \{x \in \mathbb{S} \mid A \in f_\Sigma(x)\}$.

Lemma (Exhausting lemma)

Let f_Σ be a coherent trace of a frame Σ, and suppose that there is a conditional for f_Σ which is active. Then, there is a frame Σ' and a coherent trace $f_{\Sigma'}$ extending f_Σ, such that this conditional is exhausted for $f_{\Sigma'}$.

Decidability

Definition

An *MQC-frame* for $\mathcal{L}(MQ)^{\mathcal{P}}$ is a tuple $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{K}(\mathbb{S}), \mathcal{P})$, where:

1. $\mathbb{S}$ is a set containing a subset $\mathcal{D} = \{+\alpha, -\alpha, +\beta, -\beta, +\gamma, -\gamma\}$ of designated elements (milestones).
2. $<$ is a binary relation on $\mathbb{S}$ which is transitive and connected. Moreover, $\xi \not< \xi$ for the milestones $\xi \in \mathcal{D}$.
3. $\mathcal{K}(\mathbb{S}) = \{K_1, K_2, \ldots, K_n\}$ is a partition of $\mathbb{S}$ such that:
    1. For all $x, y \in \mathbb{S}$ and $K_i \in \mathcal{K}(\mathbb{S})$, if $x, y \in K_i$, then x, y belong to the same qualitative class defined by the milestones.
    2. $\text{INF} \in \mathcal{K}(\mathbb{S})$.
4. $\mathcal{P}\colon \mathcal{C} \longrightarrow \mathcal{K}(\mathbb{S})$ is a bijection.

An *MQC-model on* Σ is an ordered pair $\mathcal{M} = (\Sigma, h)$, where h is a *meaning function* defined as above.
The concepts of satisfiability, truth and validity of a formula in a *MQC*-model are defined as usual.

Decidability

- We will use the usual *filtration method*, showing that each formula which is satisfiable in an *MQC*-model is satisfiable also in a *finite MQC*-model with bounded size.
- In order to obtain this finite model, we will define an equivalence relation of the universe of the original (non-necessarily finite) model.
- Due to the particular features of our logic, which includes a number of constants and milestones, this equivalence relation will be based on the set of subformulas of a suitable modification A^* of the formula A.
- Given a formula A written only in terms of the primitive operators we define

$$A^* =_{def} A \wedge \bigvee_{c_i \in \mathcal{C}} c_i \wedge \bigwedge_{\xi \in \mathcal{D}} (\xi \to \overrightarrow{\Box} \neg \xi)$$

Decidability

Defining filtrations

- In what follows, we denote by Γ the set of subformulas of A^*.
- Given any *MQC*-model $\mathcal{M} = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P}, h)$ of A^* and $x, y \in \mathbb{S}$, we define $x \sim_\Gamma y$ iff $\{B \in \Gamma \colon x \in h(B)\} = \{B \in \Gamma \colon y \in h(B)\}$.
- Clearly $\sim_\Gamma$ is an equivalence relation on $\mathbb{S}$.
- So, for every $x \in \mathbb{S}$, we define $[x] = \{y \in \mathbb{S} \colon y \sim_\Gamma x\}$
- $\mathbb{S}_\Gamma$ denotes the quotient set $\mathbb{S}/{\sim_\Gamma}$,
- K_Γ denotes the set $\{[x] \in \mathbb{S}_\Gamma \mid x \in K\}$.

Decidability

Filtrations

Definition

Given A^*, Γ, and $\sim_\Gamma$ as defined above, and given a *MQC*-model $\mathcal{M} = (\mathbb{S}, \mathcal{D}, <, \mathcal{K}(\mathbb{S}), \mathcal{P}, h)$ of A^*, the Γ-*filtration of* $\mathcal{M}$ is a structure of the form $\mathcal{M}_\Gamma = (\mathbb{S}_\Gamma, \mathcal{D}_\Gamma, <_\Gamma, \mathcal{K}(\mathbb{S})_\Gamma, \mathcal{P}_\Gamma, h_\Gamma)$, where:

1. $\mathbb{S}_\Gamma = \{[x] \colon x \in \mathbb{S}\}$.
2. $\mathcal{D}_\Gamma = \{[+\alpha], [+\beta], [+\gamma], [-\alpha], [-\beta], [-\gamma]\}$.
3. $\mathcal{K}(\mathbb{S})_\Gamma = \{K_\Gamma \mid K \in \mathcal{K}(\mathbb{S})\}$.
4. $\mathcal{P}_\Gamma(c_i) = \{[x] \mid x \in h(c_i)\}$.
5. $<_\Gamma \subseteq \mathbb{S}_\Gamma \times \mathbb{S}_\Gamma$, so that for every $[x], [y] \in \mathbb{S}_\Gamma$ we have $[x] <_\Gamma [y]$ iff:
    - for every $\overrightarrow{\Box} A \in \Gamma$: if $x \in h(\overrightarrow{\Box} A)$, then $y \in h(A) \cap h(\overrightarrow{\Box} A)$;
    - for every $\overleftarrow{\Box} A \in \Gamma$: if $y \in h(\overleftarrow{\Box} A)$, then $x \in h(A) \cap h(\overleftarrow{\Box} A)$.
6. $h_\Gamma(p) = \{[x] \colon x \in h(p)\}$, for every atom $p \in \Gamma$ (if $p \notin \Gamma$, $h_\Gamma(p) = \varnothing$).
7. $h_\Gamma(\xi) = \{[\xi]\}$.
8. $h_\Gamma(c_i) = \mathcal{P}_\Gamma(c_i)$.

Decidability

First results

Proposition

Given an MQC-model $\mathcal{M}$ of A^*, the Γ-filtration of $\mathcal{M}$ has at most 2^n elements in $\mathbb{S}_\Gamma$, where n is the cardinal of Γ.

Proposition

Let $\mathcal{M}_\Gamma = (\mathbb{S}_\Gamma, \mathcal{D}_\Gamma, <_\Gamma, \mathcal{K}(\mathbb{S})_\Gamma, \mathcal{P}_\Gamma, h_\Gamma)$ be the Γ-filtration of a MQC-model $\mathcal{M} = (\mathbb{S}, \mathcal{D}, <, \mathcal{K}(\mathbb{S}), \mathcal{P}, h)$. Then, $x < y$ implies $[x] <_\Gamma [y]$ for every $x, y \in \mathbb{S}$.

Proposition

If $\mathcal{M}_\Gamma = (\mathbb{S}_\Gamma, \mathcal{D}_\Gamma, \mathcal{P}_\Gamma, <_\Gamma, h_\Gamma)$ is the Γ-filtration of an MQC-model $\mathcal{M} = (\mathbb{S}, \mathcal{D}, \mathcal{P}, <, h)$, then $\mathcal{M}_\Gamma$ is also a MQC-model.

Decidability

First results

Proposition

Let $\mathcal{M}_\Gamma = (\mathbb{S}_\Gamma, \mathcal{D}_\Gamma, <_\Gamma, \mathcal{K}(\mathbb{S})_\Gamma, \mathcal{P}_\Gamma, h_\Gamma)$ *be a* Γ*-filtration of a MQC-model* $\mathcal{M} = (\mathbb{S}, \mathcal{D}, <, \mathcal{K}(\mathbb{S}), \mathcal{P}, h)$*. Then, for every* $A \in \Gamma$ *and for every* $x \in \mathbb{S}$*, we have:*

$$x \in h(A) \text{ if and only if } [x] \in h_\Gamma(A).$$

Theorem (Strong Finite Model Property)

Let A *be a formula of* $\mathcal{L}(MQ)^{\mathcal{P}}$*. If* A^* *is satisfiable in a MQC-model, then* A^* *is satisfiable in a finite MQC-model containing at most* 2^n *points, where* n *is the number of subformulas of* A^**.*

The previous theorem can be used to define a test for satisfiability (resp. validity) of a formula A with respect to *MQC*-models.

Decidability

First results

Now, the following result states the equivalence between validity wrt *MQC*-models, validity wrt *MQ*-models, and theoremhood wrt $MQ^{\mathcal{P}}$.

Theorem

For every formula A of $\mathcal{L}(MQ)^{\mathcal{P}}$*, the following conditions are equivalent:*

(i) *A is a theorem of* $MQ^{\mathcal{P}}$.

(ii) *A is MQC-valid.*

(iii) *A is MQ-valid.*

As a consequence of the previous results we obtain the following result:

Theorem (Decidability)

$MQ^{\mathcal{P}}$ *is decidable.*