Minimal solutions in Fuzzy Relation Equations. Application to Fuzzy Logic Programming

Jesús Medina Moreno

UCA Universidad de Cádiz
Departamento de Matemáticas

Department of Mathematics
University of Cádiz, Spain
jesus.medina@uca.es

A Coruña, February 24th 2015

Outline

Introduction I

- Multi-adjoint logic programming [Medina et al(2001)] is a general logical framework whose semantic structure is the multi-adjoint lattice
- Adjoint triples [Cornejo et al(2013), Medina et al(2004)] are a generalization of the t-norms and their residuated implications, which satisfy their main properties.
- They are used as the basic operators to make the calculus in several frameworks, which provides them more flexible.
- MALP, fuzzy concept lattices, fuzzy rough sets, fuzzy relation equations, etc.

Introduction II

- Fuzzy relation equations, introduced by E. Sanchez, are associated with the composition of fuzzy relations.
- FRE have been used to investigate theoretical and applicational aspects of fuzzy set theory, e.g., approximate reasoning, decision making, fuzzy control, etc.
- The multi-adjoint relation equations [Díaz and Medina(2013)] were presented as a generalization of the fuzzy relation equations.
- Two important problems in fuzzy logic programming is to find out the confidence factors of the rules in a program and abductive reasoning.
- This lecture describes and solves these problems in terms of multi-adjoint relation equation theory.

Adjoint triples

Assuming non-commutativity on the conjunctor, directly provides two different residuated (adjoint) implications

Definition

Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\& \colon P_1 \times P_2 \to P_3$, $\swarrow \colon P_3 \times P_2 \to P_1$, $\nwarrow \colon P_3 \times P_1 \to P_2$ be mappings, then $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to P_1, P_2, P_3 if:

- Adjoint property:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x$$

  where $x \in P_1$, $y \in P_2$ and $z \in P_3$.

Main properties of adjoint triples

- We have three different general sorts, which also provides a more flexible language for a potential user. Furthermore, few conditions are required.
- The adjoint triples play an important role in several important environments: fuzzy logic, fuzzy relation equations, fuzzy concept lattices, etc.
- More properties must be assumed in order to assure the mechanism for the calculus needed to resolve problems.

M. Cornejo, J. Medina, and E. Ramírez A comparative study of adjoint triples. *Fuzzy Sets and Systems*, 211:1–14, 2013.

T-norm and its residuated implication

Product adjoint triple

$\&_P \colon [0,1] \times [0,1] \to [0,1]$ defined as:

$$\&_P(x,y) = x \cdot y$$

Residuated implications: $\swarrow^P = \nwarrow_P\ [0,1] \times [0,1] \to [0,1]$ are defined as:

$$z \swarrow^P y \quad = \quad \min\{1, z/y\}$$

Granular adjoint triples

Granular product adjoint triple

Considering regular partitions of $[0,1]$ into several pieces: $[0,1]_5 = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. $\&_P^*\colon [0,1]_5 \times [0,1]_3 \to [0,1]_4$ defined as:

$$\&_P^*(x,y) = \frac{\lceil 4 \cdot x \cdot y \rceil}{4}$$

where $\lceil \_ \rceil$ is the ceil function ($\lceil 3.6 \rceil = 4$, $\lceil 7.1 \rceil = 8$, $\lceil 2 \rceil = 2, \dots$).
The residuated implications: $\swarrow_P^*\colon [0,1]_4 \times [0,1]_3 \to [0,1]_5$ and $\nwarrow_P^*\colon [0,1]_4 \times [0,1]_5 \to [0,1]_3$ are defined as:

$$z \swarrow_P^* y = \frac{\lfloor 5 \cdot \min\{1, z/y\} \rfloor}{5} \qquad z \nwarrow_P^* x = \frac{\lfloor 3 \cdot \min\{1, z/x\} \rfloor}{3}$$

where $\lfloor \_ \rfloor$ is the floor function.

Non-commutative adjoint triple

$\&: [0,1] \times [0,1] \to [0,1]$ defined as:

$$\&(x, y) = x^2 y$$

The residuated implications:
$\swarrow: [0,1] \times [0,1] \to [0,1]$ and $\nwarrow: [0,1] \times [0,1] \to [0,1]$ are defined as:

$$\begin{aligned} z \swarrow y &= \min\{1, \sqrt{z/y}\} \\ z \nwarrow x &= \min\{1, z/x\} \end{aligned}$$

Fuzzy logic

- There exists a big interest in the development of logics for dealing with information which might be either vague or uncertain.
- Several different approaches to the so-called inexact or fuzzy or approximate reasoning have been proposed, such that fuzzy, annotated, probabilistic and similarity-based logic programming.

Logic programming

Standard Logic Programming Rule *[Kowalski and van Emden]*:

$$paper\_accepted \leftarrow good\_work, good\_referees$$

Quantitative Deduction Rule *[van Emden]*:

$$paper\_accepted \overset{0.9}{\longleftarrow} good\_work \ \& \ good\_referees$$

Fuzzy Logic Programming *[Vojtáš and Paulík]*:

$$paper\_accepted \xleftarrow[product]{0.9} \min(good\_work, good\_referees)$$

Logic programming

Probabilistic Deductive Databases *[Lakshmanan and Sadri]*:

$$\left(paper\_accepted \xleftarrow{\langle[0.7,0.95],[0.03,0.2]\rangle} good\_work, good\_referees; ind, pc \right)$$

Hybrid Probabilistic Logic Programs *[Dekhtyar and Subrahmanian]*:

$$(paper\_accepted \vee_{pc} go\_conference) : [0.85, 0.98] \longleftarrow$$
$$(good\_work \wedge_{ind} good\_referees) : [0.7, 0.9] \ \& \ have\_money : [0.9, 1.0]$$

Multi-Adjoint Logic Programming

Multi-adjoint logic programming was introduced by J. Medina, M. Ojeda-Aciego and P. Vojtáš (2001) as a generalization of the previous frameworks. Among its distinctive features we emphasize:

- It is possible to use a number of different type of connectives in the rules of the programs.
- The requirements on the lattice of truth-values and on the connectives are weaker than those on other approaches.
- Sufficient conditions for continuity of the consequence operator are known.
- Completeness theorem for the computational model.

Language

A language $\mathcal{L}$ is considered, which contains propositional variables, constants, and a set of logical connectives (adjoint triples and a number of aggregators).

The language $\mathcal{L}$ is interpreted on a *(biresiduated) multi-adjoint lattice*, $(L_1, L_2, L_3, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$, where $(L_1, \preceq_1)$,$(L_2, \preceq_2)$, $(L_3, \preceq_3)$ are complete lattices and $(\&_i, \swarrow^i, \nwarrow_i)$ is a collection of adjoint triples.

Multi-adjoint logic program

A *rule* is a formula $A \swarrow^i \mathcal{B}$ or $A \nwarrow_i \mathcal{B}$,
where A is a propositional symbol (the *head*)
and $\mathcal{B}$ (the *body*) is a formula built from propositional symbols $B_1, \ldots, B_n$, and conjunctions, disjunctions and aggregations of $\mathcal{L}$.

A *multi-adjoint logic program* is a set of pairs $\langle \mathcal{R}, \alpha \rangle$,
where $\mathcal{R}$ is a rule and α is a value, which may express the confidence which the user of the system has in the truth of the rule $\mathcal{R}$.

Example: behavior of a motor

Example

The set of variables (propositional symbols)

$$\Pi = \{\mathtt{rm}, \mathtt{nb}, \mathtt{oh}, \mathtt{hfc}, \mathtt{lo}, \mathtt{lw}\}$$

The multi-adjoint lattice

$$([0,1]_{100}, [0,1]_{8}, [0,1]_{20}, \&_{\mathrm{G}}^{*}, \swarrow_{\mathrm{G}}^{*}, \nwarrow_{\mathrm{G}}^{*}, \&_{\mathrm{P}}^{*}, \swarrow_{\mathrm{P}}^{*}, \nwarrow_{\mathrm{P}}^{*}, \wedge_{\mathrm{L}})$$

The multi-adjoint program:

$$\begin{array}{lll} \langle \mathtt{hfc} & \nwarrow_{\mathrm{G}}^{*} & \mathtt{rm} \wedge_{\mathrm{L}} \mathtt{lo}, 0.75\rangle \\ \langle \mathtt{oh} & \nwarrow_{\mathrm{G}}^{*} & \mathtt{lo}, 0.5\rangle \\ \langle \mathtt{nb} & \nwarrow_{\mathrm{P}}^{*} & \mathtt{rm}, 0.75\rangle \\ \langle \mathtt{oh} & \nwarrow_{\mathrm{P}}^{*} & \mathtt{lw}, 1\rangle \\ \langle \mathtt{nb} & \nwarrow_{\mathrm{G}}^{*} & \mathtt{lo}, 1\rangle \end{array}$$

Example: behavior of a motor

Example

The usual procedure is to measure the levels of "oil", "water" and "mixture" of a specific motor, after that the values for `low_oil`, `low_water` and `rich_mixture` are obtained, which are represented in the program as:

$$\langle \mathtt{lo}, 0.20\rangle \quad \langle \mathtt{lw}, 0.20\rangle \quad \langle \mathtt{rm}, 0.50\rangle$$

Finally, the values for the rest of variables are computed.
For instance, in order to attain the value for `overheating`(o, w), for a level of oil, o, and water, w, the rules $\langle \mathtt{oh} \nwarrow^{*}_{\mathrm{G}} \mathtt{lo}, \vartheta_1\rangle$ and $\langle \mathtt{oh} \nwarrow^{*}_{\mathrm{P}} \mathtt{lw}, \vartheta_2\rangle$ are considered and its value is obtained as:

$$\mathtt{oh}(o, w) = (\mathtt{lo}(o) \,\&^{*}_{\mathrm{G}}\, \vartheta_1) \vee (\mathtt{lw}(w) \,\&^{*}_{\mathrm{P}}\, \vartheta_2)$$

Example: behavior of a motor

Example

From another point of view, the problem could be:
given the levels of oil, $o_1, \ldots, o_n$, the levels of water, $w_1, \ldots, w_n$, and the measures of mixture, $r_1, \ldots, r_n$,
and the values of the variables: $\mathtt{nb}(r_i, o_i)$, $\mathtt{hfc}(r_i, o_i)$ and $\mathtt{oh}(o_i, w_i)$, for all $i \in \{1, \ldots, n\}$;
to look for the values of ϑ_1 and ϑ_2, which solve the following system obtained after assuming the experimental data for the propositional symbols, $ov_1, o_1, w_1, \ldots, ov_n, o_n, w_n$.

$$\begin{array}{ccc}
\mathtt{oh}(ov_1) & = & (\mathtt{lo}(o_1) \,\&_{\mathrm{G}}^{*}\, \vartheta_1) \vee (\mathtt{lw}(w_1) \,\&_{\mathrm{P}}^{*}\, \vartheta_2) \\
\vdots & \vdots & \vdots \qquad\qquad \vdots \\
\mathtt{oh}(ov_n) & = & (\mathtt{lo}(o_n) \,\&_{\mathrm{G}}^{*}\, \vartheta_1) \vee (\mathtt{lw}(w_n) \,\&_{\mathrm{P}}^{*}\, \vartheta_2)
\end{array}$$

Multi-adjoint relation equations

Multi-adjoint relation equations arise as a generalization of the usual fuzzy relation equations, following the philosophy of multi-adjoint framework.

Given the universes U, V and W,
the fuzzy relations $K\colon W \times U \to P$, and $D\colon W \times V \to L_1$,
an unknown fuzzy relation $R\colon U \times V \to L_2$,
and a mapping that relates each element in U to one adjoint triple,
$\sigma\colon U \to \{1, \dots, l\}$,
a *multi-adjoint relation equation* is

$$\bigvee_{u \in U} (K(w,u) \,\&_{\sigma(u)}\, R(u,v)) = D(w,v), \quad w \in W, v \in V \qquad (1)$$

Example: behavior of a motor

Example

$U = \{\mathtt{rm}, \mathtt{lo}, \mathtt{lw}, \mathtt{rm} \wedge_{\mathrm{L}} \mathtt{lo}\}$, $V = \{\mathtt{hfc}, \mathtt{nb}, \mathtt{oh}\}$, $W = \{1, 2, 3\}$;
the mapping σ that relates the elements $\mathtt{lo}$, $\mathtt{rm} \wedge_{\mathrm{L}} \mathtt{lo}$ to the Gödel triple, and $\mathtt{rm}$, $\mathtt{lw}$ to the product triple;
and the relations $K\colon W \times U \to [0,1]_{100}$, and
$D\colon W \times V \to [0,1]_{20}$.
The unknown fuzzy relation $R\colon U \times V \to [0,1]_8$ is formed by the weights of the rules in the program.
For instance, for $v = \mathtt{oh}$,

$$\begin{aligned}
\mathtt{oh}(ov_1) &= (\mathtt{lo}(o_1) \,\&_{\mathrm{G}}^{*}\, \vartheta_{\mathtt{lo}}^{\mathtt{oh}}) \vee (\mathtt{lw}(w_1) \,\&_{\mathrm{P}}^{*}\, \vartheta_{\mathtt{lw}}^{\mathtt{oh}}) \\
\mathtt{oh}(ov_2) &= (\mathtt{lo}(o_2) \,\&_{\mathrm{G}}^{*}\, \vartheta_{\mathtt{lo}}^{\mathtt{oh}}) \vee (\mathtt{lw}(w_2) \,\&_{\mathrm{P}}^{*}\, \vartheta_{\mathtt{lw}}^{\mathtt{oh}}) \\
\mathtt{oh}(ov_3) &= (\mathtt{lo}(o_3) \,\&_{\mathrm{G}}^{*}\, \vartheta_{\mathtt{lo}}^{\mathtt{oh}}) \vee (\mathtt{lw}(w_3) \,\&_{\mathrm{P}}^{*}\, \vartheta_{\mathtt{lw}}^{\mathtt{oh}})
\end{aligned}$$

where $\vartheta_{\mathtt{lo}}^{\mathtt{oh}}$ and $\vartheta_{\mathtt{lw}}^{\mathtt{oh}}$ are the weights associated with the rules with head $\mathtt{oh}$.

The greatest solution of MARE

Given a multi-adjoint relation equation, *its associated multi-adjoint property-oriented context* is (W, U, K, σ) , and the concept lattice associated with this context will be called $\mathcal{M}_{\Pi N}(K)$.

Theorem

Let $v \in V$ and the fuzzy subset $f_v \in L_1^W$, defined as
$f_v(w) = D(w, v)$*, for all $w \in W$.*
Then the corresponding System can be solved if and only if $\langle f_v^{\downarrow^N}, f_v \rangle$ *is a concept of* $\mathcal{M}_{\Pi N}(K)$.
In this case, $f_v^{\downarrow^N}$ *is the greatest solution.*

Concept-forming operators

Given a frame $(L_1, L_2, P, \&_1, \dots, \&_l)$ and context (A, B, R, σ), we consider ${}^{\uparrow_\pi}\colon L_2^B \to L_1^A$, ${}^{\downarrow^N}\colon L_2^A \to L_1^B$:

$$\begin{aligned} g^{\uparrow_\pi}(a) &= \sup\{R(a,b) \,\&_{\sigma(b)}\, g(b) \mid b \in B\} \\ f^{\downarrow^N}(b) &= \inf\{f(a) \nwarrow_{\sigma(b)} R(a,b) \mid a \in A\} \end{aligned}$$

These definitions are generalizations of the classical and fuzzy possibility and necessity operators by Düntsch, Gediga, Georgescu, Popescu, Lai, etc.

The pair $({}^{\uparrow_\pi}, {}^{\downarrow^N})$ is an isotone Galois connection, that is ${}^{\uparrow_\pi}$ and ${}^{\downarrow^N}$ are order-preserving; and they satisfy that $f^{\downarrow^N\uparrow_\pi} \preceq_1 f$, for all $f \in L_1^A$, and that $g \preceq_2 g^{\uparrow_\pi\downarrow^N}$, for all $g \in L_2^B$.

Multi-adjoint property-oriented concept lattice

Concept

A pair of fuzzy sets $\langle g, f\rangle$, with $g \in L_2^B, f \in L_1^A$, such that $g^{\uparrow_\pi} = f$ and $f^{\downarrow^N} = g$, is called *multi-adjoint property-oriented concept*.
g is called the *extension* and f, the *intension* of the concept.

The set of the concepts

$$\mathcal{M}_{\pi N} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow_\pi} = f, f^{\downarrow^N} = g\}$$

together with the ordering $\preceq$ defined by
$\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ iff $g_1 \preceq_2 g_2$ (or $f_1 \preceq_1 f_2$) forms a complete lattice, $(\mathcal{M}_{\pi N}, \preceq)$, which is called *multi-adjoint property-oriented concept lattice*.

Example: behavior of a motor

Example

For example, the experimental data could be:

$$\texttt{oh}(ov_1) = 0.3 \quad \texttt{oh}(ov_2) = 0.6 \quad \texttt{oh}(ov_3) = 0.5$$
$$\texttt{lo}(o_1) = 0.3 \quad \texttt{lo}(o_2) = 0.6 \quad \texttt{lo}(o_3) = 0.5$$
$$\texttt{lw}(w_1) = 0.3 \quad \texttt{lw}(w_2) = 0.8 \quad \texttt{lw}(w_3) = 0.2$$

The multi-adjoint property-oriented context is (W, U, K, σ), where the relation $K\colon W \times U \to [0,1]_{100}$ is defined by

Table: Relation K.

	lo	lw
1	0.3	0.3
2	0.6	0.8
3	0.5	0.2

Example: behavior of a motor

Example

The fuzzy subset $f_{\texttt{oh}}\colon W \to [0,1]_{20}$ associated with oh is defined by $f_{\texttt{oh}}(1) = 0.3$, $f_{\texttt{oh}}(2) = 0.6$, and $f_{\texttt{oh}}(3) = 0.5$.
First of all, we compute $(f_{\texttt{oh}})^{\downarrow^N}$.

$$\begin{aligned} (f_{\texttt{oh}})^{\downarrow^N}(\texttt{lo}) &= 1.00 \\ (f_{\texttt{oh}})^{\downarrow^N}(\texttt{lw}) &= 0.75 \end{aligned}$$

And then, the fuzzy subset $(f_{\texttt{oh}})^{\downarrow^N\uparrow_\pi}$ is obtained.

$$\begin{aligned} (f_{\texttt{oh}})^{\downarrow^N\uparrow_\pi}(1) &= 0.3 \\ (f_{\texttt{oh}})^{\downarrow^N\uparrow_\pi}(2) &= 0.6 \\ (f_{\texttt{oh}})^{\downarrow^N\uparrow_\pi}(3) &= 0.5 \end{aligned}$$

Thus, the largest values to $\vartheta^{\texttt{oh}}_{\texttt{lo}}$, $\vartheta^{\texttt{oh}}_{\texttt{lw}}$ are 1.00, 0.75, respectively.

Computing the complete set of solutions

The set of solutions Fuzzy Relation Equations can be characterized, providing a useful mechanism to obtain the whole set of solutions. This characterization is given by the equivalence classes $\downarrow_N^{-1}(f_v)$.

Theorem

The whole set of solutions of System (1) is

$$\mathrm{SS}_{\&}(f_v) = (f_v^{\downarrow^N}] \setminus \bigcup\{(f_v^{\downarrow^N -}] \mid \langle f_v^{\downarrow^N -}, f_v^{-}\rangle \in \mathrm{Pre}(\langle f_v^{\downarrow^N}, f_v\rangle)\}$$

Example

The considered equation can be solved and $f_v^{\downarrow^N} = (1.000, 0.750)$ is the greatest solution.

Now, we apply Theorem 9 to obtain the set of solutions. First of all, we compute the predecessors concepts of $\langle f_v^{\downarrow^N}, f_v \rangle$ in the lattice $\mathcal{M}_{\Pi N}(K)$.

Figure: Concept lattice $\mathcal{M}_{\Pi N}$

The whole set of solutions

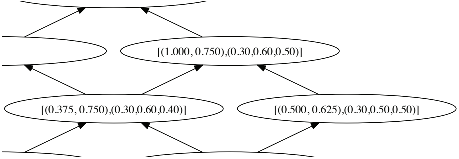

Figure: Concept lattice $\mathcal{M}_{\Pi N}$

The set of the predecessors of the greatest solution is

$$\{\langle(0.375, 0.750), (0.30, 0.60, 0.40)\rangle, \langle(0.500, 0.625), (0.30, 0.50, 0.50)\rangle\}$$

Solutions: $((1.000, 0.750)] \setminus ((0.375, 0.750)] \cup ((0.500, 0.625)]$

$$\begin{array}{lllll}
\{(1.000, 0.000), & (1.000, 0.125), & (1.000, 0.250), & (1.000, 0.375), & (1.000, 0.500), \\
(1.000, 0.625), & (1.000, 0.750), & (0.875, 0.000), & (0.875, 0.125), & (0.875, 0.250), \\
(0.875, 0.375), & (0.875, 0.500), & (0.875, 0625), & (0.875, 0.750), & (0.750, 0.000), \\
(0.750, 0.125), & (0.750, 0.250), & (0.750, 0.375), & (0.750, 0.500), & (0.750, 0.625), \\
(0.750, 0.750), & \mathbf{(0.625, 0.000)}, & (0.625, 0.125), & (0.625, 0.250), & (0.625, 0.375), \\
(0.625, 0.500), & (0.625, 0.625), & (0.625, 0.750), & \mathbf{(0.500, 0.750)}\} &
\end{array}$$

Example with no minimal solution

Frame: $([0,1]_{10}, [0,1], [0,1], \&^*_{\mathrm{G}})$.

$$\langle \texttt{oh} \quad \nwarrow^*_{\mathrm{G}} \quad \texttt{lo}, \vartheta^{\texttt{oh}}_{\texttt{lo}} \rangle$$
$$\langle \texttt{oh} \quad \nwarrow^*_{\mathrm{G}} \quad \texttt{lw}, \vartheta^{\texttt{oh}}_{\texttt{lw}} \rangle$$

Universes $U = \{\texttt{lo}, \texttt{lw}\}$, $V = \{\texttt{oh}\}$, $W = \{1, 2\}$, and the fuzzy relations K, D, defined by the matrices:

Table: Relations K and D.

	lo	lw
1	0.2	0.3
2	0.5	0.7

	oh
1	0.1
2	0.1

the equation $K \odot_\sigma R = D$ can be solved $(\langle f_v^{\downarrow^N}, f_v \rangle \in \mathcal{M}_{\Pi N}(K))$.

The greatest solution

Therefore, the greatest solution of the equation $K \odot_\sigma R = D$, which is equivalent to the system

$$\begin{aligned} 0.2 \,\&_G^* \, R(u_1, v) \vee 0.3 \,\&_G^* \, R(u_2, v) &= 0.1 \\ 0.5 \,\&_G^* \, R(u_1, v) \vee 0.7 \,\&_G^* \, R(u_2, v) &= 0.1 \end{aligned}$$

is the fuzzy relation $R \colon U \times V \to [0, 1]$, defined by $R(u_1, v) = 0.1$, $R(u_2, v) = 0.1$, which we can write as $R = (0.1, 0.1)$.

In order to find out the rest of the solutions of the system, we need to obtain the predecessors of $\langle f_v^{\downarrow^N}, f_v \rangle = \langle (0.1, 0.1), (0.1, 0.1) \rangle$.

The whole set of solutions has not minimal solutions

$$\mathrm{Pre}(\langle f_v^{\downarrow^N}, f_v \rangle) = \{\langle (0,0),(0,0)\rangle\}$$

Therefore, the complete set of solutions is

$$\begin{aligned}\mathrm{SS}_{\&}(f_v) &= (f_v^{\downarrow^N}] \setminus \bigcup \{(f_v^{\downarrow^N -}] \mid \langle f_v^{\downarrow^N -}, f_v^- \rangle \in \mathrm{Pre}(\langle f_v^{\downarrow^N}, f_v \rangle)\} \\ &= \{(x,y) \in [0,1]\times[0,1] \mid x \leq 0.1, y \leq 0.1\} \setminus \{(0,0)\} \\ &= [0,0.1]\times[0,0.1] \setminus \{(0,0)\}\end{aligned}$$

The whole set of solutions is formed by $R\colon U \times V \to [0,1]$, defined as $R(u_1,v) = x$, $R(u_2,v) = y$, with $(x,y) \in [0,0.1]\times[0,0.1] \setminus \{(0,0)\}$, which clearly has no minimal elements.

The abduction problem

The abduction problem considers two subsets of variables,
the *observed variables*, OV,
and the *hypotheses*, H,
and consists in find out the values of the hypotheses in order to explain the given values of the observed variables.

Solving the abduction problem

Example

We consider as observed variables the propositional symbols:

$$OV \quad = \quad \{\mathtt{nb}, \mathtt{oh}\}$$

and as hypotheses: $H = \{\mathtt{rm}, \mathtt{lo}, \mathtt{lw}\}$.
Hence, we know the weights of the rules and the values of their heads for an observation ov_1, nb_i and we need to find out the values of the propositional symbols in the body of each rule.
Therefore, we must solve the system of multi-adjoint relation equations:

$$\mathtt{oh}(ov_i) = (\mathtt{lo}(o_i) \,\&_{\mathrm{G}}^{*}\, \vartheta^{\mathtt{oh}}_{\mathtt{lo}}) \vee (\mathtt{lw}(w_i) \,\&_{\mathrm{P}}^{*}\, \vartheta^{\mathtt{oh}}_{\mathtt{lw}}) \vee (\mathtt{rm}(r_i) \,\&_{\mathrm{P}}^{*}\, 0)$$
$$\mathtt{nb}(nb_i) = (\mathtt{lo}(o_i) \,\&_{\mathrm{G}}^{*}\, \vartheta^{\mathtt{nb}}_{\mathtt{lo}}) \vee (\mathtt{lw}(w_i) \,\&_{\mathrm{P}}^{*}\, 0) \vee (\mathtt{rm}(r_i) \,\&_{\mathrm{P}}^{*}\, \vartheta^{\mathtt{nb}}_{\mathtt{rm}})$$

the values of $\mathtt{lo}(o_i), \mathtt{lw}(w_i), \mathtt{rm}(r_i)$ are unknown.

Example abduction reasoning

Frame: $([0,1]_{10}, [0,1], [0,1], \&_{\mathrm{G}}^{*})$.

$$\langle \mathtt{oh} \quad \swarrow_{\mathrm{G}}^{*} \quad \mathtt{lo}, 0, 2\rangle \qquad \langle \mathtt{oh} \quad \swarrow_{\mathrm{G}}^{*} \quad \mathtt{lw}, 0.3\rangle$$
$$\langle \mathtt{nb} \quad \swarrow_{\mathrm{G}}^{*} \quad \mathtt{rm}, 0.5\rangle \qquad \langle \mathtt{nb} \quad \swarrow_{\mathrm{G}}^{*} \quad \mathtt{lo}, 0.7\rangle$$

Universes $U = \{\vartheta_{\mathtt{lo}}, \vartheta_{\mathtt{lw}}\}$, $V = \{1\}$, $W = \{\mathtt{oh}, \mathtt{nb}\}$, and the fuzzy relations K, D, defined by the matrices:

Table: Relations K and D.

	$\vartheta_{\mathtt{lo}}$	$\vartheta_{\mathtt{lw}}$
oh	0.2	0.3
nb	0.5	0.7

	1
oh	0.1
nb	0.1

the equation $K \odot_{\sigma} R = D$ can be solved.

Papers

Jesús Medina and Juan Carlos Díaz-Moreno Multi-adjoint relation equations: Definition, properties and solutions using concept lattices. *Information Sciences*, 253: 100–109, 2013.

Jesús Medina and Juan Carlos Díaz-Moreno Using concept lattice theory to obtain the set of solutions of multi-adjoint relation equations. *Information Sciences*, 266: 218–225, 2014.

Jesús Medina, Esko Turunen and Juan Carlos Díaz-Moreno An algebraic characterization to compute minimal solutions of general fuzzy relation equations on linear carriers. Submitted.

Conclusions and future work

- Multi-adjoint logic programming is a general framework of fuzzy logic programming.
- Multi-adjoint relation equations are the most flexible relation equation that can be solved, at the moment.
- Two important problems in fuzzy logic programming have been considered, find out the weights of the rules of a multi-adjoint logic program and the abduction problem, which have been solved using fuzzy relation equations.
- In the future, more problems will be considered. Moreover, the comparison of other mechanism to solve the adductive reasoning will be studied.

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- Medina J, Ojeda-Aciego M, Valverde A, Vojtáš P (2004) Towards biresiduated multi-adjoint logic programming. Lecture Notes in Artificial Intelligence 3040:608–617

THANK YOU FOR YOUR ATTENTION

Jesús Medina Moreno

Universidad de Cádiz
Departamento de Matemáticas

Department of Mathematics
University of Cádiz, Spain
jesus.medina@uca.es

A Coruña, February 24th 2015