On the complexity of Temporal Equilibrium Logic

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Temporal Equilibrium Logic (TEL)

[Cabalar and Vega 2007]

- Answering Set Programming (ASP) capabilities + temporal features of standard LTL.
- For temporal reasoning not representable in ASP.
- Temporal extension of propositional Equilibrium Logic [Pearce 1996], the latter
  - well-known logical foundation of ASP;
  - generalizes stables models of ASP for arbitrary propositional theories.
- Non-monotonic semantics: selection among the models of the monotonic Temporal logic of Here-and-There (THT).

$$\mathsf{THT} = \mathsf{LTL} + \text{intuitionistic logic of Here-and-There } (\mathsf{HT})$$

Temporal logic of Here and There (THT)

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \mathsf{X}\varphi \mid \varphi \mathsf{U}\, \varphi \mid \varphi \mathsf{R}\, \varphi \qquad p \in P$$

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Derived modalities:

$\neg\varphi := \varphi \rightarrow \bot$ (negation expressed in terms of implication)
$\top := \neg\bot$
$\mathsf{F}\varphi := \top \mathsf{U}\, \varphi$ (eventually)
$\mathsf{G}\varphi := \bot \mathsf{R}\, \varphi$ (always)

THT semantics

LTL interpretation: infinite word over 2^P

THT interpretation: (H, T) such that $\mathsf{H} \sqsubseteq \mathsf{T}$

'There' LTL-interpretation

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$\mathsf{H} \sqsubseteq \mathsf{T}$ means $\mathsf{H}(i) \subseteq \mathsf{T}(i)$ for all $i \geq 0$

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(H,T) is total if $\mathsf{H} = \mathsf{T}$

THT semantics

$$\mathsf{M} = (\mathsf{H}, \mathsf{T})$$

$$
\begin{array}{ll}
\mathsf{M}, i \not\models \bot & \\
\mathsf{M}, i \models p & \Leftrightarrow p \in \mathsf{H}(i) \\
\mathsf{M}, i \models \varphi \vee \psi & \Leftrightarrow \text{either } \mathsf{M}, i \models \varphi \text{ or } \mathsf{M}, i \models \psi \\
\mathsf{M}, i \models \varphi \wedge \psi & \Leftrightarrow \mathsf{M}, i \models \varphi \text{ and } \mathsf{M}, i \models \psi \\
\mathsf{M}, i \models \varphi \rightarrow \psi & \Leftrightarrow \forall\, \mathsf{H}' \in \{\mathsf{H}, \mathsf{T}\}, \text{either } (\mathsf{H}', \mathsf{T}), i \not\models \varphi \text{ or } (\mathsf{H}', \mathsf{T}), i \models \psi \\
\mathsf{M}, i \models \mathsf{X}\varphi & \Leftrightarrow \mathsf{M}, i+1 \models \varphi \\
\mathsf{M}, i \models \varphi \,\mathsf{U}\, \psi & \Leftrightarrow \exists\, j \geq i,\ \mathsf{M}, j \models \psi \text{ and } \forall\, i \leq k < j,\ \mathsf{M}, k \models \varphi \\
\mathsf{M}, i \models \varphi \,\mathsf{R}\, \psi & \Leftrightarrow \forall\, j \geq i,\ \text{either } \mathsf{M}, j \models \psi \text{ or } \exists\, i \leq k < j,\ \mathsf{M}, k \models \varphi
\end{array}
$$

M is a THT model of φ if $\mathsf{M}, 0 \models \varphi$

THT basic properties

$$(\mathsf{H},\mathsf{T}),i \not\models \varphi \quad \not\Rightarrow \quad (\mathsf{H},\mathsf{T}),i \models \neg\varphi$$

$$(\mathsf{H},\mathsf{T}),i \models \varphi \quad \Rightarrow \quad (\mathsf{T},\mathsf{T}),i \models \varphi$$

$$(\mathsf{T},\mathsf{T}) \models \varphi \quad \Leftrightarrow \quad \mathsf{T} \models_{\mathsf{LTL}} \varphi$$

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- Dual temporal modalities independent one from the other one

$$(\mathsf{H},\mathsf{T}),i \models \mathsf{F}\varphi \quad \not\Leftrightarrow \quad (\mathsf{H},\mathsf{T}),i \models \neg\mathsf{G}\neg\varphi$$

$$(\mathsf{H},\mathsf{T}),i \models \psi\,\mathsf{U}\,\varphi \quad \not\Leftrightarrow \quad (\mathsf{H},\mathsf{T}),i \models \neg(\neg\psi\,\mathsf{R}\,\neg\varphi)$$

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$$(\mathsf{H},\mathsf{T}), i \models \psi \mathsf{U} \varphi \quad \not\Leftrightarrow \quad (\mathsf{H},\mathsf{T}), i \models \neg(\neg\psi \mathsf{R} \neg\varphi)$$

- THT satisfiability is PSPACE-complete [Cabalar and Demri 2011] (the same complexity as LTL satisfiability [Sistla and Clarke 1985]).

Temporal Equilibrium Logic (TEL)

Non-monotonic semantics: restriction of THT to a subclass of models

A TEL model of φ is a total THT model (T, T) of φ
such that
$\mathsf{H} \sqsubset \mathsf{T}$ implies $(\mathsf{H}, \mathsf{T}) \not\models \varphi$

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- TEL models: temporal generalization of stable models in propositional ASP.
  Negation interpreted as default negation in logic programs.

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- TEL models: temporal generalization of stable models in propositional ASP.
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$$\mathsf{G}(\neg p \rightarrow \mathsf{X}p)$$

Time 0, $\neg p \rightarrow \mathsf{X}p$: p false by default, $\mathsf{X}p$ holds.
Time 1, p and $\neg p \rightarrow \mathsf{X}p$: p true.
Time 2, $\neg p \rightarrow \mathsf{X}p$: ...

The unique TEL model is (T, T) where $\mathsf{T} = \emptyset, \{p\}, \emptyset, \{p\}, \ldots$

Non-existence of TEL models

LTL satisfiability $\not\Rightarrow$ TEL satisfiability

Non-existence of TEL models

$$\text{LTL satisfiability} \not\Rightarrow \text{TEL satisfiability}$$

- Use of nested implication:
  (necessary for non-existence of stable models in Equilibrium Logic)

$$\mathsf{G}(\neg p \rightarrow p)$$

$\mathsf{T} = \{p\}^\omega$ unique LTL model, but $(\emptyset^\omega, \{p\}^\omega)$ is a THT model.

Non-existence of TEL models

$$\mathsf{LTL} \text{ satisfiability} \not\Rightarrow \mathsf{TEL} \text{ satisfiability}$$

- Use of nested implication:
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  $$\mathsf{G}(\neg p \to p)$$

  $\mathsf{T} = \{p\}^\omega$ unique LTL model, but $(\emptyset^\omega, \{p\}^\omega)$ is a THT model.

- No finite justification for minimal knowledge:

  $$\mathsf{GF}\, p$$

  LTL/THT satisfiable but no TEL model.

Investigated problems

- Complexity of TEL satisfiability.
  - Systematic analysis of natural THT fragments:

bound on implication nesting depth

$$\mathsf{THT}_k^m(O_1, O_2, \ldots)$$

allowed temporal operators

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- Complexity of minimal LTL satisfiability.

  An LTL model T of φ is *minimal* if $\mathsf{H} \not\models_{\mathsf{LTL}} \varphi$ for all $\mathsf{H} \sqsubset \mathsf{T}$.

EXPSPACE-completeness for TEL satisfiability

TEL satisfiability is known to be in EXPSPACE [Cabalar and Demri 2011].

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Theorem (EXPSPACE lower bounds)

TEL satisfiability is EXPSPACE-complete even for the fragments

$$\mathsf{THT}_k^1(\mathsf{F}, \mathsf{G}, \ldots)$$

$$\mathsf{THT}_k^m(\mathsf{G}, \ldots)$$

$$\mathsf{THT}_k^m(\mathsf{U}, \ldots)$$

$m \geq 2$ (implication nesting depth) and $k \geq 2$ (temporal nesting depth)

EXPSPACE-hardness for $\mathsf{THT}_2^1(\mathsf{F}, \mathsf{G})$ is surprising because

- LTL/THT satisfiability of $\mathsf{THT}(\mathsf{F}, \mathsf{G})$ is NP-complete
  [Sistla and Clarke 1985, Cabalar and Demri 2011]
- Checking equilibrium models for HT^1 formulas is NP-complete.

EXPSPACE-hardness for $\mathsf{THT}^1_2(\mathsf{F}, \mathsf{G})$ (no nesting of implication)

Polynomial-time reduction from a domino tiling problem for grids with exponential number of columns.

$$\mathcal{I} = \langle C, \Delta, n, d_{init}, d_{final} \rangle$$

set of colors

domino types

set of domino types: tuples of four colors

Tilings of $\mathcal{I}$: grids with 2^n columns and k rows (for some k) such that

- Each cell contains a domino type;
- the first cell contains d_{init};
- the last cell is the unique one containing d_{final};
- adjacent cells have the same color on the shared edge.

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We construct $\varphi_{\mathcal{I}} \in \mathsf{THT}^1_2(\mathsf{F}, \mathsf{G})$ such that

$$\varphi_{\mathcal{I}} \text{ is } \mathsf{TEL} \text{ satisfiable } \Leftrightarrow \text{ there is a tiling of } \mathcal{I}$$

EXPSPACE-hardness for $\mathsf{THT}^1_2(\mathsf{F},\mathsf{G})$ (no nesting of implication)

Encoding of tilings of $\mathcal{I}$:

$$P_{MAIN} = \Delta \cup [1,n] \times \{0,1\} \cup \{\$\}$$

- Cells with content $d \in \Delta$ and column number $i \in [0, 2^n - 1]$ encoded by finite words in

$$\{d\}^+\{(1,b_1)\}^+ \ldots \{(n,b_n)\}^+$$

  $b_1, \ldots, b_n$ is the binary encoding of column number i.

- Tilings encoded by finite words over P_{MAIN} listing the encodings of rows from left to right, separated by occurrences of \$.

EXPSPACE-hardness for $\mathsf{THT}^1_2(\mathsf{F}, \mathsf{G})$ (no nesting of implication)

We construct $\varphi_{\mathcal{I}}$ over $P = P_{MAIN} \cup P_{TAG} \cup \{u\}$

$$\varphi_{\mathcal{I}} = \varphi_{PTC} \wedge (u \vee \varphi_{bad})$$

φ_{PTC} captures the pseudo-tiling codes (PTC) (H, T):

- T and H agree on P_{MAIN} and for all i, $\mathsf{T}(i) \cap P_{MAIN}$ is a singleton;
- *either* $\mathsf{T}(i) \supseteq P_{TAG} \cup \{u\}$ for all i ((H, T) is good),
  or $u \notin \mathsf{T}(0)$ and $\mathsf{T}(i) \cap P_{TAG}$ is a singleton;
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(T,T) is non-good: there is non-total PTC (H,T) s.t. H and T agree on $P \setminus \{u\}$.

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Since φ_{bad} is over $P \setminus \{u\}$.
Remark: every TEL model of $\varphi_{\mathcal{I}}$ is a good PTC.

EXPSPACE-hardness for $\mathsf{THT}_2^1(\mathsf{F}, \mathsf{G})$ (no nesting of implication)

$$\varphi_{\mathcal{I}} = \varphi_{PTC} \wedge (u \vee \varphi_{bad})$$

- for a good total PTC (T, T), no prefix of T encodes a tiling $\Leftrightarrow$ there is non-total PTC (H, T) satisfying φ_{bad}.

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  - tag propositions mark local portions of H: for checking that a bad condition is satisfied.
  - goodness of (T,T) is crucial for ensuring the for each bad condition B in T, there is a non-total PTC (H,T) witnessing B.

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  - tag propositions mark local portions of H: for checking that a bad condition is satisfied.
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Lemma

The TEL models of $\varphi_{\mathcal{I}}$ are the total good PTC (T,T) such that some prefix of T encodes a tiling of $\mathcal{I}$.

$\varphi_{\mathcal{I}}$ is TEL satisfiable $\Leftrightarrow$ there is a tiling of $\mathcal{I}$

Remaining main THT fragments

- Using only temporal modalities in $\{\mathsf{X}, \mathsf{F}\}$.
- No nesting of temporal modalities.
- No nesting of implication.

Using only temporal modalities X and F

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(T, T) is almost-empty if (*) for some i and for all $k \geq i$, $\mathsf{T}(k) = \emptyset$.
The *size* of (T, T) is the smallest i satisfying (*).

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Theorem (Small size model property for $\mathsf{THT}(\mathsf{X}, \mathsf{F})$)

$\varphi \in \mathsf{THT}(\mathsf{X}, \mathsf{F})$, φ is TEL satisfiable $\Rightarrow$ φ has an almost-empty TEL model of size at most $|\varphi|^3$.

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Corollary

TEL satisfiability of $\mathsf{THT}(\mathsf{X}, \mathsf{F})$ is Σ_2-complete. ☺

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Theorem (Small size model property for THT(X, F))

$\varphi \in$ THT(X, F), φ is TEL satisfiable $\Rightarrow$ φ has an almost-empty TEL model of size at most $|\varphi|^3$.

Corollary

TEL satisfiability of THT(X, F) is Σ_2-complete. ☺

- LTL/THT satisfiability of THT(X, F) is already PSPACE-complete [Sistla et Clarke 1985, Cabalar et Demri 2011]

For THT(X, F), LTL/THT satisfiability is harder than TEL satisfiability!

No nesting of temporal modalities

- LTL/THT satisfiability of THT_1 is NP-complete
  [Demri et al. 2002, Cabalar et al. 2007]
- TEL satisfiability of THT_1 is NEXPTIME-complete

No nesting of temporal modalities

- $\mathsf{LTL}/\mathsf{THT}$ satisfiability of THT_1 is NP-complete
  [Demri et al. 2002, Cabalar et al. 2007]
- TEL satisfiability of THT_1 is $\mathsf{NEXPTIME}$-complete
  - Untractable fragments of THT_1: ☹

$$\left.\begin{array}{l} \mathsf{THT}_1^m(\mathsf{F}, \mathsf{G}, \ldots) \\ \mathsf{THT}_1^m(\mathsf{U}, \ldots) \\ \mathsf{THT}_1^m(\mathsf{R}, \ldots) \end{array}\right\} \text{NEXPTIME-complete}$$

$m \geq 2$ (implication nesting depth)

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$$\left.\begin{array}{l}\mathsf{THT}_1(\mathsf{X},\mathsf{F})\\ \mathsf{THT}_1(\mathsf{X},\mathsf{G})\end{array}\right\}\ \Sigma_2\text{-complete}$$

$$\mathsf{THT}_1^1 \qquad \text{NP-complete}$$

No nesting of implication (negation expressed in terms of implication)

TEL satisfiability of THT^1 is EXPSPACE-complete

- Untractable fragments of THT^1: ☹

$\mathsf{THT}^1_m(\mathsf{F}, \mathsf{G}, \ldots)$ is EXPSPACE-complete

$m \geq 2$ (temporal nesting depth)

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THT^1_1	NP-complete
$\mathsf{THT}^1(\mathsf{X}, \mathsf{R})$	PSPACE-complete
$\mathsf{THT}^1(\mathsf{X}, \mathsf{U})$	PSPACE-complete

No nesting of implication: the fragment $\mathsf{THT}^1(\mathsf{X}, \mathsf{R})$

Remark:

$$\left.\begin{array}{l} \varphi \in \mathsf{THT}^1, \\ \mathsf{T} \text{ minimal } \mathsf{LTL} \text{ model of } \varphi \end{array}\right\} \Rightarrow (\mathsf{T}, \mathsf{T}) \text{ is a } \mathsf{TEL} \text{ model of } \varphi$$

No nesting of implication: the fragment $\mathsf{THT}^1(\mathsf{X}, \mathsf{R})$

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Lemma (Main result for $\mathsf{THT}^1(\mathsf{X}, \mathsf{R})$)

An LTL satisfiable $\mathsf{THT}^1(\mathsf{X}, \mathsf{R})$ formula has a minimal LTL model.

No nesting of implication: the fragment $\mathsf{THT}^1(\mathsf{X}, \mathsf{R})$

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An LTL satisfiable $\mathsf{THT}^1(\mathsf{X}, \mathsf{R})$ formula has a minimal LTL model.

Corollary

For $\mathsf{THT}^1(\mathsf{X}, \mathsf{R})$, LTL satisfiability = TEL satisfiability.

No nesting of implication: the fragment $\mathsf{THT}^1(\mathsf{X}, \mathsf{U})$

Remark:

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Lemma (Properties of $\mathsf{THT}(X, U)$)

Every TEL model of a $\mathsf{THT}(X, U)$ formula is almost-empty.

No nesting of implication: the fragment $\mathsf{THT}^1(\mathsf{X}, \mathsf{U})$

Remark:

$$\left.\begin{array}{l}\varphi \in \mathsf{THT}^1, \\ \mathsf{T} \text{ minimal } \mathsf{LTL} \text{ model of } \varphi\end{array}\right\} \Rightarrow (\mathsf{T}, \mathsf{T}) \text{ is a } \mathsf{TEL} \text{ model of } \varphi$$

Lemma (Properties of $\mathsf{THT}(X, U)$)

Every TEL model of a $\mathsf{THT}(X, U)$ formula is almost-empty.

Corollary (Main result for $\mathsf{THT}^1(X, U)$)

Let $\varphi \in \mathsf{THT}^1(X, U)$ and $\psi = \varphi \wedge \mathsf{FG} \bigwedge_{p \in P(\varphi)} \neg p$.

$$\varphi \text{ is } \mathsf{TEL} \text{ satisfiable } \Leftrightarrow \psi \text{ is } \mathsf{LTL} \text{ satisfiable}$$

Proof: $\Rightarrow$) by the lemma above.
$\Leftarrow$) if ψ is LTL satisfiable, then φ has a minimal LTL model. By the remark above, φ has a TEL model.

No use of implication: the fragment THT^0

Remark: every THT^0 formula is LTL and THT satisfiable.

Theorem (Lower bound for THT^0)

TEL satisfiability of THT^0 is PSPACE-hard.

Open question: the exact complexity of TEL satisfiability for THT^0.

Minimal LTL satisfiability

Theorem

Minimal LTL satisfiability is EXPSPACE-complete.

Proof: **Lower bound:** the same reduction for the lower bound of TEL satisfiability of $\mathsf{THT}^1_2(\mathsf{F}, \mathsf{G})$.

Upper bound: generalization of automata-theoretic approach for LTL satisfiability.

Minimal LTL satisfiability

Theorem

Minimal LTL satisfiability is EXPSPACE-complete.

Proof: **Lower bound:** the same reduction for the lower bound of TEL satisfiability of $\mathsf{THT}_2^1(\mathsf{F}, \mathsf{G})$.
Upper bound: generalization of automata-theoretic approach for LTL satisfiability.

- Minimal LTL satisfiability versus TEL satisfiability: different costs for THT fragments.
  - Example: for THT_1, minimal LTL satisfiability is NP-complete, while TEL satisfiability is NEXPTIME-complete.

Discussion: wrap up

- Systematic analysis of complexity of TEL satisfiability for natural THT fragments.
  - No difference between implication (resp., temporal) nesting depth 2 and $k > 2$.
  - $\mathsf{THT}(\mathsf{X}, \mathsf{F})$: the unique tractable fragment with both nesting of implication and nesting of temporal modalities.
  - Different computational cost of dual temporal modalities. Example: for $\mathsf{THT}(\mathsf{G})$, $\mathsf{EXPSPACE}$-completeness; for $\mathsf{THT}(\mathsf{X}, \mathsf{F})$, Σ_2-completeness.

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- Complexity of minimal LTL satisfiability.
  - LTL over *finite* words: LTL satisfiability = minimal LTL satisfiability.
  - LTL over infinite words: minimal LTL satisfiability exponentially harder than LTL satisfiability.

Discussion: perspectives

- Expressiveness issues for TEL fragments:
  - Kind of temporal problems expressible in tractable fragments.
  - Is the syntactical hierarchy of considered THT fragments semantically strict w.r.t. THT or TEL semantics?

    Known results: the hierarchy of $\mathsf{THT}_m(\mathsf{U})$ fragments is strict w.r.t. LTL semantics [Etessami et Wilke 1996].

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- Characterization of TEL languages:
  - Known results: TEL languages are ω-regular [Cabalar et Demri 2011].
  - Conjecture: TEL languages are LTL definable!

MANY THANKS ☺