Temporal Equilibrium Logic with Past Operators

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Abstract. In this paper we study the introduction of modal past temporal operators in Temporal Equilibrium Logic, an hybrid formalism that mixes linear-time modalities and logic programs interpreted under stable models and their characterisation in terms of Equilibrium Logic.

1 Introduction

Many scenarios of commonsense reasoning require the combination of two central dimensions in Knowledge Representation (KR): Temporal Reasoning and Non-Monotonic Reasoning (NMR). Due to the strong connection between NMR and Logic Programming (LP), one interesting possibility for this aim is to rely on the literature on (temporal) modal extensions of LP. This research area dates back to the eighties with several approaches (see survey [1]), starting with the seminal work by Luis Fariñas del Cerro [2] on the MOLOG system, that introduced different types of modalities into Prolog. Other extensions [3–5] were specifically focused on enriching LP with temporal modalities as those handled in *Linear-time Temporal Logic* (LTL) [6, 7]: $\square$ standing for "always," $\Diamond$ standing for "eventually" or $\bigcirc$ standing for "next." However, most of them imposed some syntactic restrictions and disregarded the use of default negation. For instance, the system TEMPLOG [4] introduced a particular syntax where temporal modalities could be used in the rule bodies or as the general scope of the rule conditional in a restricted manner. An example of a TEMPLOG rule is:

$$\square(p \leftarrow \bigcirc q \wedge \Diamond r) \tag{1}$$

As shown in [8], this syntax had the semantic advantage of yielding a unique, least Herbrand model for any TEMPLOG program, as happens with positive logic programs in the non-temporal case. Unfortunately, these syntactic limitations make this type of formalisms not suitable for our original purposes in KR. On the one hand, the absence of default negation is a serious drawback that prevents the representation of defaults and NMR. On the other hand, even if we focus on the temporal perspective, the way in which modal operators are used in TEMPLOG is not natural in terms of a commonsense description of a dynamic domain. Take the rule (1) as an example. This expression may make sense under a top-down Prolog reading: at any moment, to fulfil goal p we need to satisfy q at the next state and r at some point in the future. However, if we use a bottom-up reading, more common in causal laws used in action languages, (1) would assert that if q holds at the next state and r occurs in a future situation, then p is always caused to be true now. What makes an expression of this kind look unnatural is that, excepting in science fiction scripts[1], commonsense causal laws normally describe the cause-effect relations from past to future, not the other way around. For instance, if we want to express that pushing a button lights a lamp in the next situation, unless we can prove that it is broken, we would require a rule like:

$$\Box(\bigcirc light \leftarrow push \wedge \neg broken) \tag{2}$$

which cannot be represented in TEMPLOG, since it does not allow rule heads with $\bigcirc$ or $\Diamond$ operators, and cannot deal with default negation $\neg$.

Part of the syntactic limitations present in the temporal LP approaches from the eighties were mostly due to the fact that a satisfactory semantics for default negation in (non-temporal) LP was not successfully proposed until the last part of the decade. In 1988, Gelfond and Lifschitz [10] defined the *stable model* semantics that eventually gave rise to a new LP paradigm called *Answer Set Programming* (ASP) [11, 12], becoming nowadays one of the most successful frameworks for practical problem solving and KR. Moreover, as shown in [13], stable models can be logically characterised in terms of *Equilibrium Logic* [13], a formalism that defines a model selection criterion for the (monotonic) intermediate logic of *Here-and-There* (HT) [14]. The equilibrium logic characterisation has eventually allowed the definition of stable models [15] for arbitrary theories without syntactic limitations.

In principle, the extension of HT to the temporal case could be designed as an example of intuitionistic (or intermediate) modal approach, as in [16] and also studied by Luis Fariñas and Andrés Raggio in [17]. In order to obtain a temporal extension of Equilibrium Logic, one further needs to generalise the model minimisation from the latter to temporal (intuitionistic) interpretations. Such an extension of Equilibrium Logic to incorporate LTL modal operators was, in fact, proposed in a series of papers [18–20] under the name of *Temporal Equilibrium Logic* (TEL). TEL defines a temporal stable model semantics for any arbitrary theory, and so, it allows free combinations of the temporal operators and LP constructs. In this way, (2) is now representable in TEL and behaves like a standard ASP program containing the rules:

$$light(I+1) \leftarrow push(I), \neg broken(I)$$

for any integer $I \geq 0$. Moreover, we can represent other expressions that are not representable in ASP (unless we add auxiliary atoms) such as:

$$\Box(\bigcirc\Diamond light \leftarrow push \wedge \neg broken)$$

[1] As an interesting formal classification of time-travel narratives, see [9].

meaning this time that the light will be eventually on, but perhaps with a delay of $n \geq 1$ of situations. Although a prover has been built [21] to compute temporal stable models for arbitrary (propositional) temporal theories, such a syntactic flexibility is not so exploited in practice. If we look at the usual encoding of action scenarios in TEL, rules like (1) (directed from future to past) simply do not occur. In fact, this has led to the definition of a particular syntactic subset, called *splittable* [22] temporal logic programs, where formulas are constraints like $\Box(\bot \leftarrow \varphi)$ or have the form:

$$\Box(\bigcirc\alpha \leftarrow \bigcirc\beta \wedge \gamma) \tag{3}$$

where α is a disjunction of literals (an atom or its negation), and β and γ are conjunctions of literals. For instance, (2) is in splittable form. The temporal stable models for a splittable program can always be represented as LTL models of another temporal theory[2] and so can be computed [23] using an LTL model checker as a backend.

Splittable programs cover most examples of transition-based action domains in the literature and allow an arbitrary use of temporal operators in constraints. However, their expressiveness for describing causal laws is limited to (3) where the use of temporal operators is rather restrictive. Moreover, as discussed before, even if they are extended to allow more expressive operators in the rule body, as in (1), the expressions we obtain seem awkward because they would describe causation from future to past.

A more natural choice for handling expressive modalities in causal laws would be *using past operators* in the rule bodies (that express the law precondition) and using future operators for the rule heads or for the constraints describing the valid narratives. As an example, suppose that the lamp takes a pair of situations to "warm up" if we pushed the button for the first time:

$$\Box(\bigcirc\bigcirc light \leftarrow push \wedge \boxminus\neg push) \tag{4}$$

where $\boxminus$ stands for "it has always been true." Of course, we can represent this example without past operators if we introduce an auxiliary predicate to remember that *push* has been true before:

$$\begin{aligned}
&\Box(\bigcirc\bigcirc light \leftarrow push \wedge \neg pushed)\\
&\Box(\bigcirc pushed \leftarrow push)\\
&\Box(\bigcirc pushed \leftarrow pushed)
\end{aligned}$$

However, in the general case, past operators allow much more flexible and compact queries on the past narrative without the need of introducing auxiliary atoms, which may become a potential source of errors in the specification.

Another justification for the introduction of past operators relies on the fact that recent implementations of ASP solvers for incremental [24] and stream reasoning [25] which allow multiple-shot execution of the solver, can exploit

[2] It is still unknown whether this property also holds for any TEL theory or not.

the search done in previous shots if the time variable in the rules refers to the current instant in the head and previous instants in the body. For instance, for this purpose, we would rather be interested in representing (4) as the equivalent formula:

$$\Box(light \leftarrow \ominus\ominus(push \wedge \boxminus\neg pushed))$$

where $\ominus$ means "in the previous state."

It has been proved [26, 27] that LTL with past operators it can be translated into an equivalent pure future formula evaluated at the beginning of the path. Still, as shown in [28], any LTL with past is exponentially more succinct[3] than pure-future LTL.

In this paper, we consider an extension of TEL (and THT) to include past operators and show that this extension can be reduced to pure future TEL by a translation that introduces auxiliary atoms.

2 Temporal Equilibrium Logic with past operators

2.1 Syntax

The logic of *Linear Temporal Here-and-There* (THT) is defined as follows. We start from a finite set of atoms $\mathcal{L}_V$ called the *propositional signature*. The syntax of THT is the one from propositional LTL which we recall below. A temporal formula φ is defined as:

$$\begin{aligned}\varphi ::= {} & \bot \mid p \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \rightarrow \varphi_2 \mid \bigcirc\varphi_1 \mid \Box\varphi_1 \mid \Diamond\varphi_1 \mid \varphi_1 \; \mathcal{U} \; \varphi_2 \mid \varphi_1 \; \mathcal{R} \; \varphi_2 \mid \\ & \widehat{\ominus}\varphi_1 \mid \ominus\varphi_1 \mid \boxminus\varphi_1 \mid \diamondminus\varphi_1 \mid \varphi_1 \mathcal{S} \; \varphi_2 \mid \varphi_1 \mathcal{T} \; \varphi_2\end{aligned}$$

where φ_1 and φ_2 are temporal formulas in their turn and p is any atom. Negation is defined as $\neg\varphi \stackrel{\text{def}}{=} \varphi \rightarrow \bot$ whereas $\top \stackrel{\text{def}}{=} \neg\bot$. Note that '$\neg$' will stand for *default negation* in all non-monotonic formalisms described in this paper. Concerning to temporal modalities, the operators can be defined in terms of $\mathcal{U}$, $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$:

$$\begin{aligned}\Diamond\varphi &\stackrel{\text{def}}{=} \top \; \mathcal{U} \; \varphi & \Box\varphi &\stackrel{\text{def}}{=} \bot \; \mathcal{R} \; \varphi \\ \diamondminus\varphi &\stackrel{\text{def}}{=} \top \; \mathcal{S} \; \varphi & \boxminus\varphi &\stackrel{\text{def}}{=} \bot \; \mathcal{T} \; \varphi\end{aligned}$$

Operator $\Box$ is read "forever" and $\Diamond$ stands for "eventually" or "at some future point." We define the following notation for a finite concatenation of $\bigcirc$'s and $\ominus$'s operators as follows:

$$\begin{aligned}\bigcirc^0\varphi &\stackrel{\text{def}}{=} \varphi & \bigcirc^i\varphi &\stackrel{\text{def}}{=} \bigcirc(\bigcirc^{i-1}\varphi) \quad (\text{with } i \geq 1) \\ \ominus^0\varphi &\stackrel{\text{def}}{=} \varphi & \ominus^i\varphi &\stackrel{\text{def}}{=} \ominus(\ominus^{i-1}\varphi) \quad (\text{with } i \geq 1)\end{aligned}$$

[3] Assuming that no auxiliary atoms are introduced.

2.2 Semantics

An *LTL-interpretation* is an infinite sequence of sets of atoms $H_0, H_1, \ldots$ with $H_i \subseteq At$, $i \geq 0$. Given two LTL-interpretations $\mathbf{H}$ and $\mathbf{T}$, we write $\mathbf{H} \leq \mathbf{T}$ to stand for $H_i \subseteq T_i$ for all $i \geq 0$. As usual, $\mathbf{H} < \mathbf{T}$ represents $\mathbf{H} \leq \mathbf{T}$ and $\mathbf{H} \neq \mathbf{T}$, that is, the inclusion relation holds in all states but is strict $H_j \subset T_j$ for some $j \geq 0$. A *THT-interpretation* $\mathbf{M}$ is a pair of LTL-interpretations $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$, respectively standing for *here* and *there*, such that $\mathbf{H} \leq \mathbf{T}$. An interpretation $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ is said to be *total* when $\mathbf{H} = \mathbf{T}$.

Definition 1 (THT-Satisfaction). *We say that an interpretation* $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ *satisfies a formula* φ *at state* $k \in \mathbb{N}$*, written* $\mathbf{M}, k \models \varphi$*, when the following recursive conditions hold:*

1. $\mathbf{M}, k \models p$ *iff* $p \in H_k$*, for any* $p \in At$.
2. $\mathbf{M}, k \models \varphi \wedge \psi$ *iff* $\mathbf{M}, k \models \varphi$ *and* $\mathbf{M}, k \models \psi$.
3. $\mathbf{M}, k \models \varphi \vee \psi$ *iff* $\mathbf{M}, k \models \varphi$ *or* $\mathbf{M}, k \models \psi$.
4. $\mathbf{M}, k \models \varphi \rightarrow \psi$ *iff for all* $\mathbf{H}' \in \{\mathbf{H}, \mathbf{T}\}$, $\langle \mathbf{H}', \mathbf{T} \rangle, k \not\models \varphi$ *or* $\langle \mathbf{H}', \mathbf{T} \rangle, k \models \psi$.
5. $\mathbf{M}, k \models \bigcirc\varphi$ *iff* $\mathbf{M}, k+1 \models \varphi$.
6. $\mathbf{M}, k \models \widehat{\ominus}\varphi$ *iff* $\begin{cases} \mathbf{M}, k-1 \models \varphi & \textit{if } k > 0 \\ \textit{false} & \textit{if } k = 0 \end{cases}$
7. $\mathbf{M}, k \models \ominus\varphi$ *iff* $\begin{cases} \mathbf{M}, k-1 \models \varphi & \textit{if } k > 0 \\ \textit{true} & \textit{if } k = 0 \end{cases}$
8. $\mathbf{M}, k \models \varphi \;\mathcal{U}\; \psi$ *iff there is* $j \geq k$ *s.t.* $\mathbf{M}, j \models \psi$ *and* $\mathbf{M}, i \models \varphi$ *for all* $i, k \leq i < j$.
9. $\mathbf{M}, k \models \varphi \;\mathcal{R}\; \psi$ *iff for all* $j \geq k$ *s.t.* $\mathbf{M}, j \models \psi$ *or* $\mathbf{M}, i \models \varphi$ *for some* $i, k \leq i < j$.
10. $\mathbf{M}, k \models \varphi \;\mathcal{S}\; \psi$ *iff there is* j*,* $0 \leq j \leq k$ *s.t.* $\mathbf{M}, j \models \psi$ *and* $\mathbf{M}, i \models \varphi$ *for all* $i, j < i \leq k$.
11. $\mathbf{M}, k \models \varphi \;\mathcal{T}\; \psi$ *iff for all* j*,* $0 \leq j \leq k$ *s.t.* $\mathbf{M}, j \models \psi$ *or* $\mathbf{M}, i \models \varphi$ *for some* $i, j < i \leq k$.
12. *never* $\mathbf{M}, k \models \perp$. ⊠

In particular, the following LTL valid formulas are also THT valid:

$$\varphi \;\mathcal{U}\; \psi \leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi \;\mathcal{U}\; \psi)) \tag{5}$$

$$\varphi \;\mathcal{R}\; \psi \leftrightarrow \psi \wedge (\varphi \vee \bigcirc(\varphi \;\mathcal{R}\; \psi)) \tag{6}$$

$$\varphi \;\mathcal{S}\; \psi \leftrightarrow \psi \vee (\varphi \wedge \ominus(\varphi \;\mathcal{S}\; \psi)) \tag{7}$$

$$\varphi \;\mathcal{T}\; \psi \leftrightarrow \psi \wedge (\varphi \vee \ominus(\varphi \;\mathcal{T}\; \psi)) \tag{8}$$

A formula φ is THT-*valid* if $\mathbf{M}, 0 \models \varphi$ for any $\mathbf{M}$. An interpretation $\mathbf{M}$ is a THT-*model* of a theory Γ, written $\mathbf{M} \models \Gamma$, if $\mathbf{M}, 0 \models \varphi$, for all formula $\varphi \in \Gamma$. It is not difficult to see that THT-satisfaction for a total interpretation $\langle \mathbf{T}, \mathbf{T} \rangle$ collapses to LTL-satisfaction for $\mathbf{T}$. As a result:

Observation 1 $\langle \mathbf{T}, \mathbf{T} \rangle \models \Gamma$ *in THT if and only if* $\mathbf{T} \models \Gamma$ *in LTL.* ⊠

Some total models will be said to be *in equilibrium* if they satisfy the following minimality condition in their "here" component.

Definition 2 (temporal equilibrium model). *A total THT-interpretation* $\langle \mathbf{T}, \mathbf{T} \rangle$ *is a* temporal equilibrium model *of a theory* Γ *if* $\langle \mathbf{T}, \mathbf{T} \rangle \models \Gamma$ *and there is no* $\mathbf{H} < \mathbf{T}$*, such that* $\langle \mathbf{H}, \mathbf{T} \rangle \models \Gamma$*.* ⊠

Since a temporal equilibrium model is a total model $\langle \mathbf{T}, \mathbf{T} \rangle$, by Observation 1, it corresponds to an LTL model $\mathbf{T}$ we will call *temporal stable model*.

Definition 3 (temporal stable model). *If* $\langle \mathbf{T}, \mathbf{T} \rangle$ *is a temporal equilibrium model of a theory* Γ *then* $\mathbf{T}$ *is called a* temporal stable model *of* Γ *(or TS-model, for short).* ⊠

Observation 2 *Given* $\mathbf{M} = \{\mathbf{H}, \mathbf{T}\}$ *and a pair of formulas* φ, ψ*, if* $\mathbf{M}(\varphi) = \mathbf{M}(\psi)$ *then also* $\mathbf{T}(\varphi) = \mathbf{T}(\psi)$*.* ⊠

We can alternatively represent any interpretation $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ by seeing each $m_i = \langle H_i, T_i \rangle$ as a three-valued mapping $m_i : V \to \{0, 1, 2\}$ so that, for any atom p, $m_i(p) = 0$ when $p \notin T_i$ (the atom is false), $m_i(p) = 2$ when $p \in H_i$ (the atom is true), and $m_i(p) = 1$ when $p \in T_i \setminus H_i$ (the atom is undefined). We can then define a valuation for any formula φ, written[4] $\mathbf{M}(\varphi)$, by similarly considering which formulas are satisfied by $\langle \mathbf{H}, \mathbf{T} \rangle$ (which will be assigned 2), not satisfied by $\langle \mathbf{T}, \mathbf{T} \rangle$ (which will be assigned 0) or none of the two (which will take value 1). By $\mathbf{M}_i(\varphi)$ we mean the 3-valuation of φ induced by the temporal interpretation $\mathbf{M}_i$, that is, $\mathbf{M}$ shifted i positions.

Definition 4. *From the definitions in the previous section, we can easily derive the following conditions:*

1. $\mathbf{M}_i(p) \stackrel{\text{def}}{=} m_i(p)$
2. $\mathbf{M}_i(\varphi \wedge \psi) \stackrel{\text{def}}{=} min\{\mathbf{M}_i(\varphi), \mathbf{M}_i(\psi)\}$; $\quad \mathbf{M}_i(\varphi \vee \psi) \stackrel{\text{def}}{=} max\{\mathbf{M}_i(\varphi), \mathbf{M}_i(\psi)\}$
3. $\mathbf{M}_i(\varphi \to \psi) \stackrel{\text{def}}{=} \begin{cases} 2 & \text{if } \mathbf{M}_i(\varphi) \le \mathbf{M}_i(\psi) \\ \mathbf{M}_i(\psi) & \text{otherwise} \end{cases}$
4. $\mathbf{M}_i(\bigcirc\varphi) \stackrel{\text{def}}{=} \mathbf{M}_{i+1}(\varphi)$
5. $\mathbf{M}_i(\ominus\varphi) \stackrel{\text{def}}{=} \begin{cases} \mathbf{M}_{i-1}(\varphi) & \text{if } i > 0 \\ 2 & \text{if } i = 0 \end{cases}$
6. $\mathbf{M}_i(\widehat{\ominus}\varphi) \stackrel{\text{def}}{=} \begin{cases} \mathbf{M}_{i-1}(\varphi) & \text{if } i > 0 \\ 0 & \text{if } i = 0 \end{cases}$
7. $\mathbf{M}_i(\varphi\ \mathcal{U}\ \psi) \stackrel{\text{def}}{=} max\{\ min\{\mathbf{M}_j(\psi), \mathbf{M}_k(\varphi) \mid i \le k < j\} \mid j \ge i\}$
8. $\mathbf{M}_i(\varphi\ \mathcal{R}\ \psi) \stackrel{\text{def}}{=} min\{\ max\{\mathbf{M}_j(\psi), \mathbf{M}_k(\varphi) \mid i \le k < j\} \mid j \ge i\}$
9. $\mathbf{M}_i(\varphi\ \mathcal{S}\ \psi) \stackrel{\text{def}}{=} max\{\ min\{\mathbf{M}_j(\psi), \mathbf{M}_k(\varphi) \mid j < k \le i\} \mid j \le i\}$
10. $\mathbf{M}_i(\varphi\ \mathcal{T}\ \psi) \stackrel{\text{def}}{=} min\{\ max\{\mathbf{M}_j(\psi), \mathbf{M}_k(\varphi) \mid j < k \le i\} \mid j \le i\}$

[4] We use the same name $\mathbf{M}$ for a temporal interpretation and for its induced three-valued valuation function – ambiguity is removed by the way in which it is applied (a structure or a function on formulas).

Under this alternative three-valued definition, an interpretation $\mathbf{M}$ *satisfies* a formula φ when $\mathbf{M}(\varphi) = 2$. When $\mathbf{M} = \langle \mathbf{T}, \mathbf{T} \rangle$, its induced valuation will be just written as $\mathbf{T}(\varphi)$ and obviously becomes a two-valued function, that is $\mathbf{T}(\varphi) \in \{0, 2\}$. A pair of useful observations:

Observation 3 *For any interpretation* $\mathbf{M}$*,* $\mathbf{M} \models \varphi \leftrightarrow \psi$ *iff* $\mathbf{M}(\varphi) = \mathbf{M}(\psi)$ *whereas,* $\mathbf{M} \models \Box(\varphi \leftrightarrow \psi)$ *iff for all* $i \geq 0$*,* $\mathbf{M}_i(\varphi) = \mathbf{M}_i(\psi)$*.*

Example 1 (from [29]). While in TEL we can express, for instance, that any request is eventually granted:

$$\Box\,(request \rightarrow \Diamond grant)$$

with past-time modalities, we can express that a grant should be preceeded by a request

$$\Box\,(grant \rightarrow \Diamond\!\!\!\!-\, request)$$

3 Translating TEL into Quantified Equilibrium Logic

Quantified Equilibrium Logic [30] (QEL) extends Equilibrium Logic to the first-order case. As in the propositional setting, QEL defines a selection of models among those from the monotonic logic of *Quantified Here and There* (QHT).

The definition of QHT is based on a first order language denoted by $\mathcal{L} = \langle C, F, P \rangle$, where C, F and P are three disjoint sets that represent constants, functions and predicates, respectively. Given a domain D we define the sets:

- $At_D(C, P)$ stands for all atomic instances that can be formed from $\langle C \cup D, F, P \rangle$.
- $T_D(C, F)$ all ground terms that can be obtained from $\langle C \cup D, F, P \rangle$.

A *QHT-interpretation*[5] is a tuple $\mathcal{M} = \langle (D, \sigma), I_h, I_t \rangle$ such that

- $\sigma : T_D(C, F) \rightarrow D$ is a mapping from ground terms into elements of the domain satisfying that $\sigma(d) = d$ if $d \in D$
- I_h, I_t are two sets of ground atoms from $At_D(C, P)$ such that $I_h \subseteq I_t$

Given two QHT interpretations, $\mathcal{M} = \langle (D, \sigma), I_h, I_t \rangle$ and $\mathcal{M}' = \langle (D', \sigma'), I'_h, I'_t \rangle$, we say that $\mathcal{M} \leq \mathcal{M}'$ iff $D = D'$, $\sigma = \sigma'$, $I_t = I'_t$ and $I_h \subseteq I'_h$. If, additionally, $I_h \subset I'_h$ we say that the relation is strict (denoted by $\mathcal{M} < \mathcal{M}'$).

Definition 5 (QHT semantics from [30]). *The satisfaction relation for a QHT interpretation* $\mathcal{M} = \langle (D, \sigma), I_h, I_t \rangle$ *is defined as follows:*

[5] We assume here a version of QHT taking *static* domain and *decidable* equality. Briefly, this means that the domain D is common to worlds h and t and that equality is a "decidable" predicate, that is, it satisfies the excluded middle axiom $(x = y) \vee \neg(x = y)$.

- $\mathcal{M} \models \top, \quad \mathcal{M} \not\models \bot$
- $\mathcal{M} \models p(\tau_1, \cdots, \tau_n)$ *iff* $p(\sigma(\tau_1), \cdots, \sigma(\tau_n)) \in I_h$
- $\mathcal{M} \models \tau = \tau'$ *iff* $\sigma(\tau) = \sigma(\tau')$.
- $\mathcal{M} \models \varphi \wedge \psi$ *iff* $\mathcal{M} \models \varphi$ *and* $\mathcal{M} \models \psi$
- $\mathcal{M} \models \varphi \vee \psi$ *iff* $\mathcal{M} \models \varphi$ *or* $\mathcal{M} \models \psi$
- $\mathcal{M} \models \varphi \to \psi$ *iff* $\mathcal{M} \not\models \varphi$ *or* $\mathcal{M} \models \psi$, *and* $\langle (D, \sigma), I_t, I_t \rangle \models \varphi \to \psi$
- $\mathcal{M} \models \forall x,\ \varphi(x)$ *iff* $\mathcal{M} \models \varphi(d)$, *for all* $d \in D$
- $\mathcal{M} \models \exists x,\ \varphi(x)$ *iff* $\mathcal{M} \models \varphi(d)$, *for some* $d \in D$ ⊠

As usual, we say that a QHT-interpretation $\mathcal{M}$ is a *model* of a first order theory Γ iff $\mathcal{M} \models \phi$ for all $\phi \in \Gamma$.

Definition 6 (quantified equilibrium model from [30]). *Let φ be a QHT formula. A QHT total interpretation $\mathcal{M}$ is a* first-order equilibrium model *of φ if $\mathcal{M} \models \varphi$ and there is no model $\mathcal{M}' < \mathcal{M}$ of φ.* ⊠

For our purposes, it is convenient to define a particular subclass of QHT theories. We define the fragment of QHT called *monadic here-and-there with inequality*, MHT($\leq$), by syntactically restricting all predicates to monadic, excepting a binary predicate $\leq$. Moreover, we also fix the domain D to be the set of natural numbers $D = \mathbb{N}$ so that $\leq$ captures the standard ordering among them. We consider the time constant 0 to stand for the initial situation. Given that both the domain and the interpretation of $\leq$ are fixed, interpretations will only vary for ground atoms in $At(\mathbb{N}, P)$, that is, those formed with the set of monadic predicates P and elements from $\mathbb{N}$. Then, MHT($\leq$) interpretations can be simply given by pairs $\langle \mathcal{H}, \mathcal{T} \rangle$ with $\mathcal{H} \subseteq \mathcal{T} \subseteq At(\mathbb{N}, P)$.

As usual, we write $x > y$ to stand for $\neg(x \leq y)$. We will also use the following abbreviations:

$$
\begin{array}{ll}
\forall x \geq t.\ \varphi \stackrel{\text{def}}{=} \forall x(t \leq x \to \varphi) & \forall x \in [t, z).\ \varphi \stackrel{\text{def}}{=} \forall x(t \leq x \wedge x < z \to \varphi) \\
\exists x \geq t.\ \varphi \stackrel{\text{def}}{=} \exists x(t \leq x \wedge \varphi) & \exists x \in [t, z).\ \varphi \stackrel{\text{def}}{=} \exists x(t \leq x \wedge x < z \wedge \varphi)
\end{array}
$$

Fragment MHT($\leq$) imposes exactly the same restrictions on QHT than the so-called *monadic first-order logic with inequality*, FOL($\leq$), does on classical First-Order Logic (FOL). This subclass of FOL was used by Kamp in his famous theorem [6] where he proved that LTL is exactly as expressive as FOL($\leq$), so that we can actually see the former as a fragment of the latter. This result was separated into two directions: proving that LTL can be translated into FOL($\leq$) and vice versa. For the first direction, Kamp defined the following translation from modal formulas into quantified first-order expressions:

Definition 7 (Kamp's translation). *Kamp's translation for a temporal formula* φ *and a timepoint* $t \in \mathbb{N}$*, denoted by* $[\varphi]_t$*, is recursively defined as follows:*

$$
\begin{aligned}
[\bot]_t &\stackrel{\text{def}}{=} \bot \\
[p]_t &\stackrel{\text{def}}{=} p(t), \ \textit{with } p \in At. \\
[\neg\alpha]_t &\stackrel{\text{def}}{=} \neg[\alpha]_t \\
[\alpha \wedge \beta]_t &\stackrel{\text{def}}{=} [\alpha]_t \wedge [\beta]_t \\
[\alpha \vee \beta]_t &\stackrel{\text{def}}{=} [\alpha]_t \vee [\beta]_t \\
[\alpha \rightarrow \beta]_t &\stackrel{\text{def}}{=} [\alpha]_t \rightarrow [\beta]_t \\
[\bigcirc\alpha]_t &\stackrel{\text{def}}{=} [\alpha]_{t+1} \\
[\alpha \ \mathcal{U} \ \beta]_t &\stackrel{\text{def}}{=} \exists x \geq t. \, ([\beta]_x \wedge \forall y \in [t, x). \, [\alpha]_y) \\
[\alpha \ \mathcal{R} \ \beta]_t &\stackrel{\text{def}}{=} \forall x \geq t. \, ([\beta]_x \vee \exists y \in [t, x). \, [\alpha]_y) \\
[\ominus\alpha]_t &\stackrel{\text{def}}{=} [\alpha]_{t-1} \\
[\alpha \ \mathcal{S} \ \beta]_t &\stackrel{\text{def}}{=} \exists \ 0 \leq x \leq t. \, ([\beta]_x \wedge \forall y \in (x, t]. \, [\alpha]_y) \\
[\alpha \ \mathcal{T} \ \beta]_t &\stackrel{\text{def}}{=} \forall \ 0 \leq x \leq t. \, ([\beta]_x \vee \exists y \in (x, t]. \, [\alpha]_y)
\end{aligned}
$$

where $[\alpha]_{t+1}$ *and* $[\alpha]_{t-1}$ *are, respectively, abbreviations of*

$$\exists y \geq t. \ \big([\alpha]_y \wedge \neg\exists z(t < z \wedge z < y)\big) \tag{9}$$

$$\exists y \leq t. \ \big([\alpha]_y \wedge \neg\exists z(y < z \wedge z < t)\big). \tag{10}$$

⊠

Note how, per each atom $p \in At$ in the temporal formula φ, we get a monadic predicate $p(x)$ in the translation.

The effect of this translation on the derived operators $\Diamond$, $\Box$, $\diamondminus$ and $\boxminus$ yields the quite natural expressions:

$$
\begin{aligned}
[\Box\alpha]_t &\equiv \forall x \geq t. \, [\alpha]_x & \qquad [\Diamond\alpha]_t &\equiv \exists x \geq t. \, [\alpha]_x \\
[\boxminus\alpha]_t &\equiv \forall x \leq t. \, [\alpha]_x & \qquad [\diamondminus\alpha]_t &\equiv \exists x \leq t. \, [\alpha]_x
\end{aligned}
$$

Definition 8 (THT-MHT(≤) interpretation correspondence). *Given a THT interpretation* $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ *on a signature* $\mathcal{L}_V$*, we say that the MHT(≤)-interpretation* $\mathcal{M} = \langle \mathcal{H}, \mathcal{T} \rangle$ corresponds *to* $\mathbf{M}$ *iff*

- $p \in H_i$ *iff* $p(i) \in \mathcal{H}$*, for all* $i \in \mathbb{N}$*.*
- $p \in T_i$ *iff* $p(i) \in \mathcal{T}$*, for all* $i \in \mathbb{N}$*.* ⊠

We now prove that when considering this model correspondence, Kamp's translation allows us to translate a THT theory into a corresponding QHT one.

Theorem 1. *Let φ be a THT formula built on a set of atoms $\mathcal{L}_V$, $\mathbf{M} = \langle \mathbf{H}, \mathbf{T} \rangle$ a THT-interpretation on $\mathcal{L}_V$ and $\mathcal{M} = \langle \mathcal{H}, \mathcal{T} \rangle$ its corresponding MHT(≤)-interpretation from Definition 8. It holds that:*

$$\forall i \in \mathbb{N},\ \mathbf{M}, i \models \varphi \textit{ iff } \mathcal{M} \models [\varphi]_i.$$

⊠

Proof. We proceed by structural induction.

- If $\varphi = \bot$ then $[\varphi]_i = \bot$ and the result is straightforward.
- If $\varphi = p$ is an atom, then $[p]_i = p(i)$ and we get the chain of equivalent conditions: $\mathbf{M}, i \models p \Leftrightarrow p \in H_i \Leftrightarrow p(i) \in \mathcal{H} \Leftrightarrow \mathcal{M} \models p(i)$.
- If $\varphi = \alpha \wedge \beta$ we get:

$$\begin{aligned} \mathbf{M}, i \models \alpha \wedge \beta &\Leftrightarrow \mathbf{M}, i \models \alpha \text{ and } \mathbf{M}, i \models \alpha \\ &\Leftrightarrow \mathcal{M} \models [\alpha]_i \text{ and } \mathcal{M} \models [\beta]_i \quad \text{(induction on } \alpha, \beta) \\ &\Leftrightarrow \mathcal{M} \models [\alpha]_i \wedge [\beta]_i \\ &\Leftrightarrow \mathcal{M} \models [\alpha \wedge \beta]_i \end{aligned}$$

- The proof for $\varphi = \alpha \vee \beta$ is analogous to the one for $\alpha \wedge \beta$.
- If $\varphi = \alpha \to \beta$ we get:

$$\mathbf{M}, i \models \alpha \to \beta \Leftrightarrow \text{ for any } w \in \{\mathbf{H}, \mathbf{T}\},\ \langle w, \mathbf{T} \rangle, i \not\models \alpha \text{ or } \langle w, \mathbf{T} \rangle, i \models \beta$$

  Now, since the THT-interpretation $\langle \mathbf{T}, \mathbf{T} \rangle$ also corresponds to the MHT(≤) interpretation $\langle \mathcal{T}, \mathcal{T} \rangle$ we can apply induction on subformulas, so that we continue with the equivalent conditions:

$$(\text{ for any } w \in \{\mathcal{H}, \mathcal{T}\},\ \langle w, \mathcal{T} \rangle \not\models [\alpha]_i \text{ or } \langle w, \mathcal{T} \rangle \models [\beta]_i) \Leftrightarrow (\langle \mathcal{H}, \mathcal{T} \rangle \models [\alpha \to \beta]_i)\,.$$

- If $\varphi = \circ\alpha$ we get the equivalent conditions:

$$\begin{aligned} \mathbf{M}, i \models \circ\alpha &\Leftrightarrow \mathbf{M}, i+1 \models \alpha \\ &\Leftrightarrow \mathcal{M} \models [\alpha]_{i+1} \quad \text{(by induction)} \\ &\Leftrightarrow \mathcal{M} \models [\circ\alpha]_i \end{aligned}$$

- If $\varphi = \ominus\alpha$ we get the equivalent conditions:

$$\begin{aligned} \mathbf{M}, i \models \ominus\alpha &\Leftrightarrow \mathbf{M}, i-1 \models \alpha \\ &\Leftrightarrow \mathcal{M} \models [\alpha]_{i-1} \quad \text{(by induction)} \\ &\Leftrightarrow \mathcal{M} \models [\ominus\alpha]_i \end{aligned}$$

- If $\varphi = \alpha\ \mathcal{U}\ \beta$ we get the equivalent conditions:

$$\begin{aligned} \mathbf{M}, i \models \alpha\ \mathcal{U}\ \beta &\Leftrightarrow \exists k \text{ s.t. } k \geq i \text{ and } \mathbf{M}, k \models \beta \text{ and } \forall j \in \{i, \ldots, k-1\},\ \mathbf{M}, j \models \alpha \\ &\Leftrightarrow \exists k \text{ s.t. } k \geq i \text{ and } \mathcal{M} \models [\beta]_k \text{ and } \forall j \in \{i, \ldots, k-1\},\ \mathcal{M} \models [\alpha]_j{}^{6} \\ &\Leftrightarrow \exists k \text{ s.t. } k \geq i \text{ and } \mathcal{M} \models [\beta]_k \text{ and } \forall j \text{ if } i \leq j < k \text{ then } \mathcal{M} \models [\alpha]_j \\ &\Leftrightarrow \mathcal{M} \models [\alpha\ \mathcal{U}\ \beta]_i. \end{aligned}$$

[6] Here we apply the induction hypothesis on α and β.

– The proof for $\varphi = \alpha\ \mathcal{R}\ \beta$ is analogous to the one for $\alpha\ \mathcal{U}\ \beta$.
– If $\varphi = \alpha\ \mathcal{S}\ \beta$ we get the equivalent conditions:

$$\begin{aligned}\mathbf{M}, i \models \alpha\ \mathcal{S}\ \beta &\Leftrightarrow \exists k \text{ s.t. } 0 \le k \le i \text{ and } \mathbf{M}, k \models \beta \text{ and } \forall j \in \{k+1, \dots, i\},\ \mathbf{M}, j \models \alpha \\ &\Leftrightarrow \exists k \text{ s.t. } 0 \le k \le i \text{ and } \mathcal{M} \models [\beta]_k \text{ and } \forall j \in \{k+1, \dots, i\},\ \mathcal{M} \models [\alpha]_j{}^{7} \\ &\Leftrightarrow \exists k \text{ s.t. } 0 \le k \le i \text{ and } \mathcal{M} \models [\beta]_k \text{ and } \forall j \ \text{ if } k < j \le i \text{ then } \mathcal{M} \models [\alpha]_j \\ &\Leftrightarrow \mathcal{M} \models [\alpha\ \mathcal{S}\ \beta]_i.\end{aligned}$$

– The proof for $\varphi = \alpha\ \mathcal{T}\ \beta$ is analogous to the one for $\alpha\ \mathcal{S}\ \beta$.

Corollary 1. *Let* $\mathbf{T}$ *be a temporal interpretation,* $\mathcal{T}$ *its corresponding first-order interpretation and* φ *some temporal formula. Then,* $\mathbf{T}$ *is a TS-model of* φ *iff* $\mathcal{T}$ *is a stable model of* $[\varphi]_0$. ⊠

4 Removing past operators

Proposition 1. *For any THT formulas* φ *and* ψ *built on the signature* $\mathcal{L}_V$*, the following formulas are tautologies en THT.*

(T. 1) $(\ominus\varphi \leftrightarrow \top)$
(T. 2) $(\widehat{\ominus}\varphi \leftrightarrow \bot)$
(T. 3) $(\varphi\mathcal{S}\psi \leftrightarrow \psi)$
(T. 4) $(\varphi\mathcal{T}\psi \leftrightarrow \psi)$
(T. 5) $\square(\bigcirc\ominus\varphi \leftrightarrow \varphi)$
(T. 6) $\square(\bigcirc(\varphi\mathcal{S}\psi) \leftrightarrow (\bigcirc\psi \vee (\bigcirc\varphi \wedge \varphi\mathcal{S}\psi)))$
(T. 7) $\square(\bigcirc(\varphi\mathcal{T}\psi) \leftrightarrow (\bigcirc\psi \wedge (\bigcirc\varphi \vee \varphi\mathcal{T}\psi)))$.

Proof. (T. 1)-(T. 4) follow directly from Definition 1, we following prove (T. 5)-(T. 7).

(T. 5)

$$\begin{aligned}\mathbf{M}_0(\square(\bigcirc\ominus\varphi \leftrightarrow \varphi)) = 2 &\Leftrightarrow \forall i \ge 0, \mathbf{M}_i(\bigcirc\ominus\varphi) = \mathbf{M}_i(\varphi) \\ &\Leftrightarrow \forall i \ge 0, \mathbf{M}_{i+1}(\ominus\varphi) = \mathbf{M}_i(\varphi) \\ &\Leftrightarrow \forall i \ge 0, \mathbf{M}_i(\varphi) = \mathbf{M}_i(\varphi) \\ &\Leftrightarrow \top\end{aligned}$$

(T. 6)

$$\begin{aligned}&\mathbf{M}_0(\square(\bigcirc(\varphi\mathcal{S}\psi) \leftrightarrow (\bigcirc\psi \vee (\bigcirc\varphi \wedge \varphi\mathcal{S}\psi)))) = 2 \\ \Leftrightarrow\ & \forall i \ge 0\ \mathbf{M}_i(\bigcirc(\varphi\mathcal{S}\psi)) = \mathbf{M}_i(\bigcirc\psi \vee (\bigcirc\varphi \wedge \varphi\mathcal{S}\psi)) \\ \Leftrightarrow\ & \forall i \ge 0\ \mathbf{M}_i(\bigcirc(\varphi\mathcal{S}\psi)) = max\{\mathbf{M}_{i+1}(\psi), min\{\mathbf{M}_{i+1}(\varphi), \mathbf{M}_i(\varphi\mathcal{S}\psi)\}\} \\ \Leftrightarrow\ & \forall i \ge 0\ \mathbf{M}_i(\bigcirc(\varphi\mathcal{S}\psi)) = max\{\mathbf{M}_{i+1}(\psi), min\{\mathbf{M}_{i+1}(\varphi), \mathbf{M}_{i+1}(\ominus(\varphi\mathcal{S}\psi))\}\} \\ \Leftrightarrow\ & \forall i \ge 0\ \mathbf{M}_i(\bigcirc(\varphi\mathcal{S}\psi)) = max\{\mathbf{M}_{i+1}(\psi), \mathbf{M}_{i+1}(\varphi \wedge \ominus(\varphi\mathcal{S}\psi))\} \\ \Leftrightarrow\ & \forall i \ge 0\ \mathbf{M}_i(\bigcirc(\varphi\mathcal{S}\psi)) = \mathbf{M}_{i+1}(\psi \vee (\varphi \wedge \ominus(\varphi\mathcal{S}\psi))).\end{aligned}$$

[7] As happens in the proof for the operator $\mathcal{U}$, this step of the proof comes from the application of the induction hypothesis on α and β.

Finally, by applying (7) we conclude:

$$\forall i \geq 0 \ \mathbf{M}_i\left(\bigcirc\left(\varphi \mathcal{S} \psi\right)\right) = \mathbf{M}_{i+1}\left(\varphi \mathcal{S} \psi\right) \Leftrightarrow \top.$$

(T. 7)

$$\begin{aligned}
&\mathbf{M}_0\left(\Box\left(\bigcirc\left(\varphi \mathcal{T} \psi\right) \leftrightarrow \left(\bigcirc\psi \wedge \left(\bigcirc\varphi \vee \varphi \mathcal{T} \psi\right)\right)\right)\right) = 2 \\
\Leftrightarrow\ &\forall i \geq 0 \ \mathbf{M}_i\left(\bigcirc\left(\varphi \mathcal{T} \psi\right)\right) = \mathbf{M}_i\left(\bigcirc\psi \wedge \left(\bigcirc\varphi \vee \varphi \mathcal{T} \psi\right)\right) \\
\Leftrightarrow\ &\forall i \geq 0 \ \mathbf{M}_i\left(\bigcirc\left(\varphi \mathcal{T} \psi\right)\right) = min\left\{\mathbf{M}_{i+1}\left(\psi\right), max\left\{\mathbf{M}_{i+1}\left(\varphi\right), \mathbf{M}_i\left(\varphi \mathcal{T} \psi\right)\right\}\right\} \\
\Leftrightarrow\ &\forall i \geq 0 \ \mathbf{M}_i\left(\bigcirc\left(\varphi \mathcal{T} \psi\right)\right) = min\left\{\mathbf{M}_{i+1}\left(\psi\right), max\left\{\mathbf{M}_{i+1}\left(\varphi\right), \mathbf{M}_{i+1}\left(\ominus\left(\varphi \mathcal{T} \psi\right)\right)\right\}\right\} \\
\Leftrightarrow\ &\forall i \geq 0 \ \mathbf{M}_i\left(\bigcirc\left(\varphi \mathcal{T} \psi\right)\right) = min\left\{\mathbf{M}_{i+1}\left(\psi\right), \mathbf{M}_{i+1}\left(\varphi \vee \ominus\left(\varphi \mathcal{T} \psi\right)\right)\right\} \\
\Leftrightarrow\ &\forall i \geq 0 \ \mathbf{M}_i\left(\bigcirc\left(\varphi \mathcal{T} \psi\right)\right) = \mathbf{M}_{i+1}\left(\psi \wedge \left(\varphi \vee \ominus\left(\varphi \mathcal{T} \psi\right)\right)\right).
\end{aligned}$$

Finally, by applying (8) we conclude:

$$\forall i \geq 0 \ \mathbf{M}_i\left(\bigcirc\left(\varphi \mathcal{T} \psi\right)\right) = \mathbf{M}_{i+1}\left(\varphi \mathcal{T} \psi\right) \Leftrightarrow \top.$$

Definition 9 (Labelling). *Let γ and χ be two THT formulas in $\mathcal{L}_V$ such that the latter is of the form $\ominus\varphi$, $\widehat{\ominus}\varphi$, $\varphi\mathcal{S}\psi$ or $\varphi\mathcal{T}\psi$. We define $\gamma^{\chi}_{\mathbf{L}_\chi}$ on $V_{\mathbf{L}} = \mathcal{L}_V \cup \{\mathbf{L}_\chi\}$, with $\mathbf{L}_\chi$ being a fresh atom, as follows:*

$$\gamma^{\chi}_{\mathbf{L}_\chi} = \begin{cases}
\bot & if\ \gamma = \bot \\
p & if\ \gamma = p \in \mathcal{L}_V \\
\circ\varphi^{\chi}_{\mathbf{L}_\chi} & if\ \gamma = \circ\varphi \\
\varphi^{\chi}_{\mathbf{L}_\chi} \odot \psi^{\chi}_{\mathbf{L}_\chi} & if\ \gamma = \varphi \odot \psi\ \ and\ \odot \in \{\wedge, \vee, \rightarrow, \mathcal{U}, \mathcal{R}\} \\
\ominus(\varphi^{\chi}_{\mathbf{L}_\chi}) & if\ \gamma = \ominus\varphi\ \ and\ \gamma \neq \chi \\
\widehat{\ominus}(\varphi^{\chi}_{\mathbf{L}_\chi}) & if\ \gamma = \widehat{\ominus}\varphi\ \ and\ \gamma \neq \chi \\
(\varphi^{\chi}_{\mathbf{L}_\chi})\ \mathcal{S}\ (\psi^{\chi}_{\mathbf{L}_\chi}) & if\ \gamma = \varphi\mathcal{S}\psi\ \ and\ \gamma \neq \chi \\
(\varphi^{\chi}_{\mathbf{L}_\chi})\ \mathcal{T}\ (\psi^{\chi}_{\mathbf{L}_\chi}) & if\ \gamma = \varphi\mathcal{T}\psi\ \ and\ \gamma \neq \chi \\
\mathbf{L}_\chi & if\ \gamma = \chi
\end{cases}$$

Broadly speaking, $\gamma^{\chi}_{\mathbf{L}_\chi}$ results from replacing every occurrence of χ by $\mathbf{L}_\chi$ in the subformulas of γ. ⊠

Definition 10. *Given a THT formula χ in $\mathcal{L}_V$, of the form $\ominus\varphi$, $\widehat{\ominus}\varphi$, $\varphi\mathcal{S}\psi$ or $\varphi\mathcal{T}\psi$ and a THT interpretation (in three-valued form) $\mathbf{M}$, we denote by $\mathbf{M}^e$ the following THT interpretation built on $V_{\mathbf{L}} = \mathcal{L}_V \cup \{\mathbf{L}_\chi\}$:*

$$\mathbf{M}^e_i(p) = \begin{cases} \mathbf{M}_i(\chi)\ if\ p = \mathbf{L}_\chi \\ \mathbf{M}_i(p)\ \ if\ p \in \mathcal{L}_V \end{cases} \tag{11}$$

⊠

With χ we define $df(\chi)$ as follows:

$$df(\chi) \stackrel{\text{def}}{=} \begin{cases} \Box(\circ\mathbf{L}_\chi \leftrightarrow \varphi) \wedge (\mathbf{L}_\chi \leftrightarrow \top) & \text{if } \gamma = \ominus\varphi; \\ \Box(\circ\mathbf{L}_\chi \leftrightarrow \varphi) \wedge (\mathbf{L}_\chi \leftrightarrow \bot) & \text{if } \gamma = \widehat{\ominus}\varphi; \\ \Box(\circ\mathbf{L}_\chi \leftrightarrow \circ\psi \vee (\circ\varphi \wedge \mathbf{L}_\chi)\ ) \wedge (\mathbf{L}_\chi \leftrightarrow \psi) & \text{if } \gamma = (\varphi\ \mathcal{S}\ \psi); \\ \Box(\circ\mathbf{L}_\chi \leftrightarrow \circ\psi \wedge (\circ\varphi \vee \mathbf{L}_\chi)\ ) \wedge (\mathbf{L}_\chi \leftrightarrow \psi) & \text{if } \gamma = (\varphi\ \mathcal{T}\ \psi). \end{cases}$$

From Proposition 1 and the definition of $\mathbf{M}^e$, it is easy to determine that $df(\chi)$ is always a tautology in $\mathbf{M}^e$.

Lemma 1. *Let γ and χ be two THT formulas in $\mathcal{L}_V$ such that the latter is of the form $\ominus\varphi$, $\widehat{\ominus}\varphi$, $\varphi\mathcal{S}\psi$ or $\varphi\mathcal{T}\psi$. If $\mathbf{M}$ is a model of γ then the THT interpretation on $V_\mathbf{L}$, $\mathbf{M}^e$ defined before satifies that $\mathbf{M} = \mathbf{M}^e \cap V$ and also:*

$$\mathbf{M}^e \models \{\gamma^\chi_{\mathbf{L}_\chi}\} \cup \{df(\chi)\}.$$

Proof. For any atom $p \in \mathcal{L}_V$ and $i \geq 0$, $\mathbf{M}^e_i(p) = \mathbf{M}_i(p)$, thus, the valuations for atoms in $\mathbf{M}$ and $\mathbf{M}^e$ coincide, which means that $\mathbf{M} = \mathbf{M}^e \cap V$. Furthermore, since γ does not have labels and $\mathbf{M} \models \gamma$, this means that

$$\mathbf{M}_0(\gamma) \stackrel{(11)}{=} \mathbf{M}^e_0(\gamma) = 2.$$

On the other hand, if $\gamma = \chi$ we get that

$$\mathbf{M}^e_0(\gamma^\chi_{\mathbf{L}_\chi}) = \mathbf{M}^e_0(\mathbf{L}_\chi) = \mathbf{M}_0(\chi) = 2.$$

To prove that $\mathbf{M}^e$ satisfies the translation, it remains to be shown that $\mathbf{M}^e \models df(\gamma)$. This proof comes directly from Proposition 1 and the fact that $\mathbf{M}^e_i(\mathbf{L}_\chi) = \mathbf{M}_i(\chi)$, for all $i \geq 0$.

Lemma 2. *Let γ be a THT formula in $\mathcal{L}_V$ and $\mathbf{M}$ a THT interpretation such that*

$$\mathbf{M}^e \models \{\gamma^\chi_{\mathbf{L}_\chi}\} \cup \{df(\chi)\}.$$

For any THT formula χ of the form $\ominus\varphi$, $\widehat{\ominus}\varphi$, $\varphi\mathcal{S}\psi$ or $\varphi\mathcal{T}\psi$ and any $i \geq 0$, the following property holds:

$$\mathbf{M}^e_i(\gamma^\chi_{\mathbf{L}_\chi}) = \mathbf{M}^e_i(\gamma).$$

Proof. We use structural induction on γ.

1. When the subformula γ has the shape $\top$, $\bot$ or an atom p this is trivial, since $\gamma^\chi_{\mathbf{L}_\chi} = \gamma$ by definition.
2. When $\gamma = \varphi \bullet \psi$ for any connective $\bullet \in \{\wedge, \vee, \rightarrow\}$, then the proof follows from Definition 4 and by applying induction on $\varphi^\chi_{\mathbf{L}_\chi}$ and $\psi^\chi_{\mathbf{L}_\chi}$.

To finish the proof, notice that $df(\chi)$ is always a tautology in $\mathbf{M}^e$.

3. When $\gamma = \bigcirc\varphi$:

$$\begin{aligned}
\mathbf{M}^e_i((\bigcirc\varphi)^\chi_{\mathbf{L}_\chi}) &= \mathbf{M}^e_i(\bigcirc\varphi^\chi_{\mathbf{L}_\chi}) \\
&= \mathbf{M}^e_{i+1}(\varphi^\chi_{\mathbf{L}_\chi}) \\
&= \mathbf{M}^e_{i+1}(\varphi) && \text{(induction)} \\
&= \mathbf{M}^e_i(\bigcirc\varphi)
\end{aligned}$$

4. $\gamma = (\varphi\ \mathcal{U}\ \psi)$: we get

$$\begin{aligned}
\mathbf{M}^e_i((\varphi\ \mathcal{U}\ \psi)^\chi_{\mathbf{L}_\chi}) &= \mathbf{M}^e_i((\varphi)^\chi_{\mathbf{L}_\chi}\ \mathcal{U}\ (\psi)^\chi_{\mathbf{L}_\chi}) \\
&= max\{min\{\mathbf{M}^e_j(\psi^\chi_{\mathbf{L}_\chi}), \mathbf{M}^e_k(\varphi^\chi_{\mathbf{L}_\chi}) \mid i \le k < j\} \mid j \ge i\} \\
&= max\{min\{\mathbf{M}^e_j(\psi), \mathbf{M}^e_k(\varphi) \mid i \le k < j\} \mid j \ge i\} && \text{(induction)} \\
&= \mathbf{M}^e_i(\varphi\mathcal{U}\psi)
\end{aligned}$$

5. The proof for $\gamma = \varphi\mathcal{R}\psi$ is similar to the one presented in the previous case.
6. $\gamma = \ominus\varphi$: here we must consider two cases; when $\gamma \neq \chi$ and $\gamma = \chi$. In the former case proceed as follows:

$$\begin{aligned}
\mathbf{M}^e_i((\ominus\varphi)^\chi_{\mathbf{L}_\chi}) &= \mathbf{M}^e_i(\ominus(\varphi)^\chi_{\mathbf{L}_\chi}) && (\gamma \neq \chi) \\
&= \mathbf{M}^e_{i-1}((\varphi)^\chi_{\mathbf{L}_\chi}) \\
&= \mathbf{M}^e_{i-1}(\varphi) && \text{(induction)} \\
&= \mathbf{M}^e_i(\ominus\varphi)
\end{aligned}$$

while if $\ominus\varphi = \chi$ and $i = 0$, we have that:

$$\begin{aligned}
\mathbf{M}^e_0((\ominus\varphi)^\chi_{\mathbf{L}_\chi}) &= \mathbf{M}^e_0(\mathbf{L}_\chi) && (\chi = \gamma) \\
&= \mathbf{M}^e_0(\top) && df(\chi) \\
&= \mathbf{M}^e_0(\ominus\varphi). && \text{((T. 1) from Prop. 1)}
\end{aligned}$$

When $i > 0$:

$$\begin{aligned}
\mathbf{M}^e_i((\ominus\varphi)^\chi_{\mathbf{L}_\chi}) &= \mathbf{M}^e_i(\mathbf{L}_\chi) && (\chi = \gamma) \\
&= \mathbf{M}^e_{i-1}(\bigcirc\mathbf{L}_\chi) \\
&= \mathbf{M}^e_{i-1}(\varphi) && df(\chi) \\
&= \mathbf{M}^e_i(\ominus\varphi) && \text{(Definition 4)}
\end{aligned}$$

7. $\gamma = \widehat{\ominus}\varphi$: following the same reasoning as above, for the cases $\gamma \neq \chi$ and $\gamma = \chi$ (with $i > 0$), the argument we used coincides with the case $\gamma = \ominus\varphi$. The proof for the remaining case, $\gamma = \chi$ and $i = 0$ is presented below:

$$\begin{aligned}
\mathbf{M}^e_0((\widehat{\ominus}\varphi)^\chi_{\mathbf{L}_\chi}) &= \mathbf{M}^e_0(\mathbf{L}_\chi) && (\chi = \gamma) \\
&= \mathbf{M}^e_0(\bot) && df(\chi) \\
&= \mathbf{M}^e_0(\widehat{\ominus}\varphi) && \text{((T. 2) from Prop. 1)}
\end{aligned}$$

8. $\gamma = \varphi\mathcal{S}\psi$: as in the previous cases, if $\gamma \neq \chi$ we have that:

$$\begin{aligned}
\mathbf{M}^e_i((\varphi\ \mathcal{S}\ \psi)^\chi_{\mathbf{L}_\chi}) &= \mathbf{M}^e_i((\varphi)^\chi_{\mathbf{L}_\chi}\ \mathcal{S}\ (\psi)^\chi_{\mathbf{L}_\chi}) && (\chi \neq \gamma) \\
&= max\{min\{\mathbf{M}^e_j(\psi^\chi_{\mathbf{L}_\chi}), \mathbf{M}^e_k(\varphi^\chi_{\mathbf{L}_\chi}) \mid j < k \le i\} \mid j \le i\} \\
&= max\{min\{\mathbf{M}^e_j(\psi), \mathbf{M}^e_k(\varphi) \mid j < k \le i\} \mid j \le i\} && \text{(induction)} \\
&= \mathbf{M}^e_i(\varphi\mathcal{S}\psi).
\end{aligned}$$

On the other hand, if $\gamma = \chi$ and $i = 0$, we have that:

$$\begin{aligned}
\mathbf{M}_0^e((\varphi\ \mathcal{S}\ \psi)^{\chi}_{\mathbf{L}_\chi}) &= \mathbf{M}_0^e(\mathbf{L}_\chi) && (\chi = \gamma)\\
&= \mathbf{M}_0^e(\psi) && df(\chi)\\
&= \mathbf{M}_0^e(\varphi\mathcal{S}\psi). && \text{((T. 3) from Prop. 1)}
\end{aligned}$$

However, in the case $i > 0$ we can only prove that:

$$\begin{aligned}
\mathbf{M}_i^e((\varphi\ \mathcal{S}\ \psi)^{\chi}_{\mathbf{L}_\chi}) &= \mathbf{M}_i^e(\mathbf{L}_\chi) && (\chi = \gamma)\\
&= \mathbf{M}_{i-1}^e(\bigcirc\mathbf{L}_\chi) &&\\
&= \mathbf{M}_{i-1}^e(\bigcirc\psi \vee (\bigcirc\varphi \wedge \mathbf{L}_\chi)) && (df(\chi))\\
&= max\{\mathbf{M}_{i-1}^e(\bigcirc\psi), min\{\mathbf{M}_{i-1}^e(\bigcirc\varphi), \mathbf{M}_{i-1}^e(\mathbf{L}_\chi)\}\} &&\\
&= max\{\mathbf{M}_i^e(\psi), min\{\mathbf{M}_i^e(\varphi), \mathbf{M}_{i-1}^e(\mathbf{L}_\chi)\}\}. &&
\end{aligned}$$

Unfortunately, we cannot get rid of $\mathbf{L}_\chi$, since χ itself is the formula to be proved in the induction step. To prove that $\mathbf{M}_i^e(\mathbf{L}_\chi) = \mathbf{M}_i^e(\chi)$, we will equivalently show that

$$\forall i \geq 0\ \mathbf{M}_i^e(\mathbf{L}_\chi) = \mathbf{M}_i^e(\varphi\mathcal{S}\psi).$$

For the base case ($i = 0$) we proceed as follows:

$$\begin{aligned}
\mathbf{M}_0^e(\mathbf{L}_\chi) &= \mathbf{M}_0^e(\psi) && (df(\chi))\\
&= \mathbf{M}_0^e(\varphi\mathcal{S}\ \psi) && \text{((T. 3) from Prop. 1).}
\end{aligned}$$

For the inductive step we have:

$$\begin{aligned}
\mathbf{M}_i^e(\mathbf{L}_\chi) &= \mathbf{M}_{i-1}^e(\bigcirc\mathbf{L}_\chi) &&\\
&= \mathbf{M}_{i-1}^e(\bigcirc\psi \vee (\bigcirc\varphi \wedge \mathbf{L}_\chi)) && (df(\chi))\\
&= max\{\mathbf{M}_i^e(\psi),\ min\{\mathbf{M}_i^e(\varphi), \mathbf{M}_{i-1}^e(\mathbf{L}_\chi)\}\} &&\\
&= max\{\mathbf{M}_i^e(\psi),\ min\{\mathbf{M}_i^e(\varphi), \mathbf{M}_{i-1}^e(\varphi\mathcal{S}\ \psi)\}\} && (Induction)\\
&= \mathbf{M}_i^e(\psi \vee (\varphi \wedge \ominus(\varphi\mathcal{S}\ \psi))) && \text{(Equivalence (7))}\\
&= \mathbf{M}_i^e(\varphi\mathcal{S}\ \psi). &&
\end{aligned}$$

9. The proof for $\gamma = \varphi\mathcal{T}\psi$ is similar to the proof for $\gamma = \varphi\mathcal{S}\psi$.

Theorem 2. *Let γ in $\mathcal{L}_V$ be a formula and χ one of its subformulas whose form is $\ominus\varphi$, $\widehat{\ominus}\varphi$, $\varphi\mathcal{S}\psi$ or $\varphi\mathcal{T}\psi$. It holds that:*

$$\{\mathbf{M} \mid \mathbf{M} \models \gamma\} = \{\mathbf{M}^e \cap V \mid \mathbf{M}^e \models \{\gamma^{\chi}_{\mathbf{L}_\chi}\} \cup \{df(\chi)\}.$$

Proof. The '$\subseteq$' inclusion immediately follows from Lemma 1. For proving the '$\supseteq$' inclusion, suppose that we have $\mathbf{M}^e$ such that $\mathbf{M}^e \models \gamma^{\chi}_{\mathbf{L}_\chi} \wedge df(\chi)$. By Lemma 2 we conclude that $\mathbf{M}_0^e(\gamma) = \mathbf{M}_0^e(\gamma^{\chi}_{\mathbf{L}_\chi}) = 2$, so $\mathbf{M}^e$ is a model of γ. Since γ is a formula in $\mathcal{L}_V$, it follows that $\mathbf{M}^e \cap V \models \gamma$. ⊠

Corollary 2. *Given a past-THT formula γ on $\mathcal{L}_V$, every past operator in γ can be removed by introducing auxiliary atoms.*

5 Conclusions

In this paper we have presented an extension of Temporal Equilibrium Logic (TEL) that introduces the use of past modalities. We have defined the syntax and semantics of this extension and provided a translation that, by introducing auxiliary atoms, allows removing past modalities and using the original version of TEL exclusively dealing with future operators.

The immediate future work will be focused on the implementation of these operators in the tool `STeLP` and the study on complexity of the current translation. We also plan to study the potential application as a high-level language for incremental ASP.

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