Strong Negation in Well-Founded and Partial Stable Semantics for Logic Programs

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Outline

1. Introduction
   - Overview of Logic Programming semantics
   - Logical foundations
   - Partial Equilibrium Logic
2. Contributions
   - Routley semantics and strong negation
   - HT^2 with strong negation
   - Partial Equilibrium Logic with strong negation
3. Conclusions

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Stable Models [Gelfond & Lifschitz 88]
Partial Stable Models [Przymusinski 91]
Well-Founded semantics (WFS) [van Gelder, Ross & Schlipf 91]

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  M stable model iff classical minimal model of Π^M

Example:

We guess some M
say $M = \{q, r\}$
to interprete $\neg\alpha$'s

$$\begin{array}{lcl} p & \leftarrow & r \wedge \neg q \\ q & \leftarrow & r \wedge \neg p \\ r & \leftarrow & \neg p \end{array}$$

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  Example:

We guess some M		Minimal model
say $M = \{q, r\}$	$q \leftarrow r$	$\{q, r\} = M$
to interprete $\neg\alpha$'s	r	stable model !

Logic programming: semantics for default negation

- Partial stable models [Przymusinski 91]
  M partial stable model iff 3-valued minimal-truth model of Π^M
  Again similar idea: reduct + fixpoint condition
- Note that interpretations are now 3-valued.
  Well-founded model = partial stable model with minimal info. (defined atoms)
- Example: $p \leftarrow \neg p$ has no stable model.
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- Default negation $\neg p$ means *no evidence on p*
  What if we want to represent p *is false* ($\sim p$)?

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- Reduct: not exactly a semantic definition.
  Syntax is restricted: no arbitrary formulas.
- Our goal: look for a logical style definition.
  Get minimal models inside some (monotonic) logic.
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  - Full logical interpretation of connectives.
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What about the second negation?

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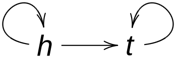
Stable models and Equilibrium Logic

- (Monotonic) intermediate logic of *here-and-there* (*HT*)

  Intuitionistic $\subseteq$ *HT* $\subseteq$ Classical



- Pearce's *Equilibrium Logic*: minimal *HT* models
  Intuition: *t* world is fixed (plays the role of "reduct"), *h* world is minimized

- Interesting results:
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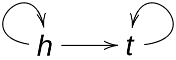

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  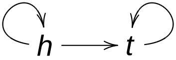

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[Cabalar,Odintsov & Pearce KR'06] *Partial* Equilibrium Logic

1. takes minimal models on monotonic logic HT^2
2. HT^2 classified inside [Došen 86] framework N combined with [Routley & Routley 72].
3. Main idea: each world

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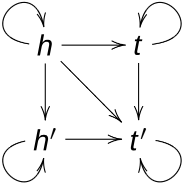

$\leq$ Accessibility relation like any intermediate logic
($w \models p$ and $w \leq w'$) implies $w' \models p$

$\leq$ used for implication: $w \models \varphi \rightarrow \psi$ when
$\forall w' \geq w$, $w' \models \varphi$ implies $w' \models \psi$

But negation $\neg\phi$ is no longer defined as $\phi \rightarrow \bot$

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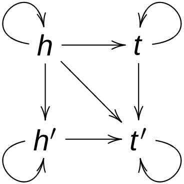

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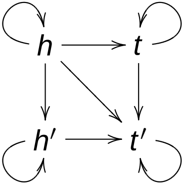

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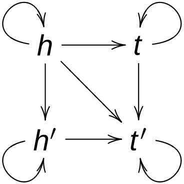

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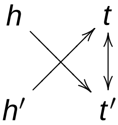

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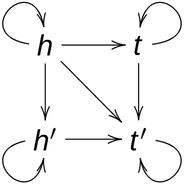

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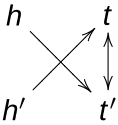

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Partial equilibrium models

- Let H, H', T, T' denote sets of atoms verified at h, h', t, t'.
- A model can be seen as a pair $\langle \mathbf{H}, \mathbf{T} \rangle$ of 3-valued interp. where $\mathbf{H} = (H, H')$ and $\mathbf{T} = (T, T')$.
- Define an ordering among models, $\langle \mathbf{H}_1, \mathbf{T}_1 \rangle \trianglelefteq \langle \mathbf{H}_2, \mathbf{T}_2 \rangle$ if:
  - (i) $\mathbf{T}_1 = \mathbf{T}_2$ (this is fixed)
  - (ii) $\mathbf{H}_1$ less truth than $\mathbf{H}_2$ ($H_1 \subseteq H_2$ and $H'_1 \subseteq H'_2$).
- $\langle \mathbf{H}, \mathbf{T} \rangle$ is said to be *total* if $\mathbf{H} = \mathbf{T}$.

Definition (Partial equilibrium model)

A model $\mathcal{M}$ of theory Π is a *partial equilibrium (PE) model* of Π if it is total and $\trianglelefteq$-minimal.

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- $\langle \mathbf{H}, \mathbf{T} \rangle$ is said to be *total* if $\mathbf{H} = \mathbf{T}$.

Definition (Partial equilibrium model)

A model $\mathcal{M}$ of theory Π is a *partial equilibrium (PE) model* of Π if it is total and $\trianglelefteq$-minimal.

Partial equilibrium models

- Let H, H', T, T' denote sets of atoms verified at h, h', t, t'.
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Theorem (Corresp. to Partial Stable Models)

For a normal or disjunctive logic program Π, $\langle \mathbf{T}, \mathbf{T} \rangle$ is a partial equilibrium model of Π iff $\mathbf{T}$ is a partial stable model of Π.

Definition (strong equivalence)

Two theories Π_1, Π_2 are said to be *strongly equivalent* if for any set of formulas Γ, $\Pi_1 \cup \Gamma$ and $\Pi_2 \cup \Gamma$ have the same partial stable models.

Theorem (from KR'06 paper)

Π_1, Π_2 *are PEL strongly equivalent iff they are equivalent in* HT^2.

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Outline

$N^{*\sim}$

- HT^2 special case of N^* family = intuitionistic Kripke frames with a weaker negation [Routley & Routley 72].
- We define next $N^{*\sim}$, adding strong negation $\sim$, as follows.
- Syntax: atoms, $\wedge$, $\vee$, $\rightarrow$, $\neg$ (weak negation) and $\sim$ (strong negation)
- Inference rules: modus ponens, plus

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$N^{*\sim}$ axioms

1. the axiom schemes of positive logic,
2. weak negation axioms:

   W1. $\neg\alpha \wedge \neg\beta \rightarrow \neg(\alpha \vee \beta)$ **W2.** $\neg(\alpha \wedge \beta) \rightarrow \neg\alpha \vee \neg\beta$
   W3. $\neg(\alpha \rightarrow \alpha) \rightarrow \beta$

   Until now, N^*
3. and for $N^{*\sim}$, we add the schemata for strong negation from [Vorob'ev 52]:

   N1. $\sim(\alpha \rightarrow \beta) \leftrightarrow \alpha \wedge \sim\beta$ **N2.** $\sim(\alpha \wedge \beta) \leftrightarrow \sim\alpha \vee \sim\beta$
   N3. $\sim(\alpha \vee \beta) \leftrightarrow \sim\alpha \wedge \sim\beta$ **N4.** $\sim\sim\alpha \leftrightarrow \alpha$
   N5. $\sim\neg\alpha \leftrightarrow \alpha$

$N^{*\sim}$ models

Definition ($N^{*\sim}$ frame)

is a triple $\langle W, \leq, * \rangle$, where:

1. W is a set of worlds
2. $\leq$ a partial order on W
3. $* : W \longrightarrow W$ such that $x \leq y$ iff $y^* \leq x^*$.

Definition ($N^{*\sim}$ model)

is an $N^{*\sim}$ frame $\langle W, \leq, *, V^+, V^- \rangle$ plus two valuations $V^+, V^- : At \times W \longrightarrow \{0,1\}$ such that:

$$V^{+(-)}(p,u) = 1 \;\;\&\;\; u \leq w \;\;\Rightarrow\;\; V^{+(-)}(p,w) = 1$$

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$N^{*\sim}$ valuation

V^+, V^- are extended to arbitrary formulas as follows:

- $V^+(\varphi \wedge \psi, w) = 1$ iff $V^+(\varphi, w) = V^+(\psi, w) = 1$
- $V^+(\varphi \vee \psi, w) = 1$ iff $V^+(\varphi, w) = 1$ or $V^+(\psi, w) = 1$
- $V^+(\varphi \rightarrow \psi, w) = 1$ iff for every w' such that $w \leq w'$, $V^+(\varphi, w') = 1 \Rightarrow V^+(\psi, w') = 1$
- $V^+(\neg\varphi, w) = 1$ iff $V^+(\varphi, w^*) = 0$
- $V^+(\sim \varphi, w) = 1$ iff $V^-(\varphi, w) = 1$

- $V^-(\varphi \wedge \psi, w) = 1$ iff $V^-(\varphi, w) = 1$ or $V^-(\psi, w) = 1$
- $V^-(\varphi \vee \psi, w) = 1$ iff $V^-(\varphi, w) = V^-(\psi, w) = 1$
- $V^-(\varphi \rightarrow \psi, w) = 1$ iff $V^+(\varphi, w) = 1$ and $V^-(\psi, w) = 1$
- $V^-(\neg\varphi, w) = 1$ iff $V^+(\varphi, w) = 1$
- $V^-(\sim \varphi, w) = 1$ iff $V^+(\varphi, w) = 1$

$N^{*\sim}$ properties

- Axiom (**W3**) allows defining an intuitionistic negation $\bot := \neg(p_0 \rightarrow p_0)$ and $-\alpha := \alpha \rightarrow \bot$

Proposition

The $\langle \vee, \wedge, \rightarrow, - \rangle$-fragment of $N^{*\sim}$ coincides with intuitionistic logic.

Proposition

$N^{*\sim}$ is a conservative extension of N^* and of Nelson's paraconsistent logic N^-.

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For each formula ϕ there exists some $N^{*\sim}$-equivalent formula ψ in negation normal form ($\sim$ only applied to atoms).

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Theorem (Vorob'ev reduction)

For each formula ϕ, let ϕ' be the result of:

1. *obtaining its negation normal form and*
2. *replacing each $\sim p$ by a new atom p'.*

Then: $N^{*\sim} \vdash \phi$ iff $N^{*\sim} \vdash \phi'$.

Outline

HT^2 with strong negation

- $HT^2 = N^* + Ax$ where Ax are more axioms for weak negation
  Nothing new is required: $HT^{2\sim} = N^{*\sim} + Ax$

The (common) set of axioms Ax is the following:

W4. $-\alpha \vee --\alpha$
W5. $-\alpha \vee (\alpha \rightarrow (\beta \vee (\beta \rightarrow (\gamma \vee -\gamma))))$
W6. $\bigwedge_{i=0}^{2}((\alpha_i \rightarrow \bigvee_{j \neq i} \alpha_j) \rightarrow \bigvee_{j \neq i} \alpha_j) \rightarrow \bigvee_{i=0}^{2} \alpha_i$
W7. $\alpha \rightarrow \neg\neg\alpha$
W8. $\alpha \wedge \neg\alpha \rightarrow \neg\beta \vee \neg\neg\beta$
W9. $\neg\alpha \wedge \neg(\alpha \rightarrow \beta) \rightarrow \neg\neg\alpha$
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W11. $\neg\neg\alpha \wedge \neg\neg\beta \rightarrow (\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$

plus the rule (EC) $\frac{\alpha \vee (\beta \wedge \neg\beta)}{\alpha}$

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- $HT^{2\sim} = HT^2 + \{\textbf{N1}, \ldots, \textbf{N5}\}$.
- $HT^{2\sim}$ frames coincide with HT^2 ones seen before:

relation $\leq$ $\quad$ $*$ function

- Note: we allow paraconsistency:
  p and $\sim p$ can be both founded.

Proposition

Vorob'ev reduction also holds for $HT^{2\sim}$.

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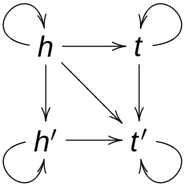

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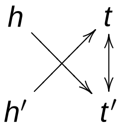

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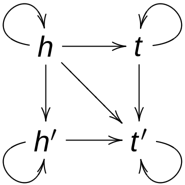

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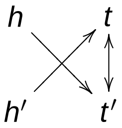

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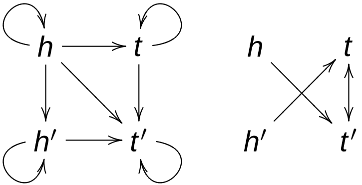

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HT^2 with strong negation

We extend HT^2 with a new truth constant u (undefinedness).

Definition (HT^2_u)

$V(u,h) = V(u,t) = 0$ and $V(u,h') = V(u,t') = 1$.
That is, always undefined.

Theorem

$HT^2_u = HT^2 + \{u \leftrightarrow \neg u\}$

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Other useful logics:

- Semi-consistency: $HT^2_{sc} := HT^{2\sim}_u + \{p \wedge \sim p \rightarrow u$
  yields the effect:

  $p, \sim p$ can be both non-unfounded, but not both founded.

- Coherence: $HT^2_{coh} := HT^{2\sim}_u + \{p \rightarrow \neg \sim p \vee u, \ \ \sim p \rightarrow \neg p \vee u\}$
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  p founded $\Rightarrow \sim p$ unfounded
  $\sim p$ founded $\Rightarrow p$ unfounded

Proposition

HT^2_{coh} (coherence) is stronger than HT^2_{sc} (semi-consistency).

- Vorob'ev reductions for these variants: just apply translation to axiom schemata too.

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PEL with strong negation

Theorem (Strong equivalence)

Let Γ_1, Γ_2 be sets of $N^{*\sim}$ formulas.
Γ_1, Γ_2 are strongly equivalent (wrt each version of PEL models)
iff
Γ_1, Γ_2 equivalent in the corresp. monotonic logic $HT_u^{2\sim}$, HT_{sc}^2, HT_{coh}^2.

Proposition

For all PEL variants with strong neg., complexity of reasoning tasks is the same class as that of ordinary PEL (in particular, decision problem is Π_2^P-hard).

PEL with strong negation

Theorem (Strong equivalence)

Let Γ_1, Γ_2 be sets of $N^{*\sim}$ formulas.
Γ_1, Γ_2 are strongly equivalent (wrt each version of PEL models)
iff
Γ_1, Γ_2 equivalent in the corresp. monotonic logic $HT_u^{2\sim}$, HT_{sc}^2, HT_{coh}^2.

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Correspondence theorems

An extended logic program Π is a set of rules r:

$$Hd(r) \leftarrow B(r)$$

where $Hd(r)$ is a literal ($p, \sim p$) and
$B(r)$ a conjunction of expressions like L or $\neg L$ (L=literal).

Theorem

$\langle \mathbf{T}, \mathbf{T} \rangle$ is an HT^2_{sc} PE model of an extended program Π iff $\mathbf{T}$ is a classical-negation [Przymusinski 91] part. stable model of Π.

Theorem

$\langle \mathbf{T}, \mathbf{T} \rangle$ is an HT^2_{coh} PE model of an extended program Π iff $\mathbf{T}$ is a strong-negation [Alferes & Pereira 92] part. stable model of Π.

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$$\begin{array}{rcl} Hd(r) & \leftarrow & B(r) \wedge u \wedge \neg \sim Hd(r) \\ Hd(r) \vee u & \leftarrow & B(r) \end{array}$$

Theorem

A pair $\mathbf{T} = (T, T')$ is a WFSX part. stable model [Alferes & Pereira 92] of an extended logic program Π

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Conclusions

- PEL is a natural logical foundation for partial stable models. Strong negation added preserving complexity and strong equivalence results.
- We provided a Routley-style general family $N^{*\sim}$ of strong neg. logics
- We explored 3 different options:
  - $HT_u^{2\sim}$ paraconsistency
  - HT_{sc}^2 semi-consistency
  - HT_{coh}^2 coherence $\sim L \Rightarrow \neg L$
- Coherence:
  - not so natural when handling paraconsistency
  - for capturing WFSX, HT_{coh}^2 is too strong
  - WFSX can be encoded into HT_{sc}^2
- Future work: detailed comparison to frame-based characterisation of WFSX [Alcântara,Damásio&Pereira].

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Further reading

- P. Cabalar, S. Odintsov & D. Pearce. Logical Foundations of Well-Founded Semantics. In *Proceedings KR 06*.

- P. Cabalar, S. Odintsov, D. Pearce & A. Valverde. Analysing and Extending Well-Founded and Partial Stable Semantics using Partial Equilibrium Logic. In *Proceedings of ICLP'06*, (LNCS 4079).
  - Strong equivalence, complexity results, properties of PEL inference, disjunctive WFS, . . .

- P. Cabalar, S. Odintsov, D. Pearce & A. Valverde. On the logic and computation of Partial Equilibrium Models. In *Proceedings of JELIA'06*, (LNAI 4160).
  - Tableaux proof system
  - Splitting theorem for PEL