A Logic for Reasoning about Well-Founded Semantics: Preliminary Report

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Abstract. The paper presents a preliminary solution to a long-standing problem in the foundations of well-founded semantics for logic programs. The problem addressed is this: which logic can be considered adequate for well-founded semantics (WFS) in the sense that its minimal models (appropriately defined) coincide with the partial stable models of a logic program? We approach this problem by identifying the HT^2 frames previously proposed by Cabalar [4] to capture WFS as structures of a kind used by Dosen [5] to characterise a family of logics weaker than intuitionistic and minimal logic. We identify partial stable models as minimal models in this semantics and we axiomatise the resulting logic.

1 Introduction

In logic programming, among the several approaches proposed for dealing with default negation that go beyond the methods of ordinary Prolog, the *stable model semantics* of Gelfond and Lifschitz [7] and the *well-founded semantics* (WFS) of Van Gelder, Ross and Schlipf [16] have proved to be the most attractive and resilient. Each forms the basis for working systems that currently have promising applications in different subareas of AI problem solving.

Although concepts from classical or many-valued logic typically appear in the respective definitions of these two approaches to logic programming with negation, they are normally characterised by fixpoint constructions quite different from the usual model-theoretic semantics found in logic. In a certain sense therefore they lack a logical foundation. One way to obtain a more logical representation is to re-describe stable, partial stable or well-founded models as minimal models in an appropriate (monotonic) logic. The problem is to identify the logic and the minimality condition. The reward is that the logic provides a conceptual foundation for the LP semantics and helps to characterise its inference relation. As an example, logically equivalent programs will clearly have the same minimal models and hence the same semantics. Another positive consequence is that, if the logic and minimality condition are general enough, then one will obtain a natural generalisation of the LP semantics to ordinary propositional or predicate logic,

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without the syntactic restrictions typically present in logic programming. How to extend the restricted syntax of so-called normal programs has in fact formed one of the principal research programmes within the field of logic programming and nonmonotonic reasoning.

A logic and minimality condition suitable for capturing stable models and answer sets was provided in [11]. The logic involved is a well-known intermediate logic, *here-and-there* (HT) [8], and the minimality condition has given rise to a nonmonotonic system of inference called *equilibrium logic*. Successive extensions of the syntax of answer set programs have agreed with equilibrium logic (ie produced equivalent semantics), so in this case the more operational and the more logical approaches have coincided. Recently, a variant of equilibrium logic has been defined that captures and hence generalises the semantics of paraconsistent answer sets [10].

Our aim in this paper is to provide a similar result for the case of WFS, which has remained open until now. A first incomplete solution to this problem was proposed in [4] where partial stable models were characterised in terms of a generalisation of HT frames called HT^2 frames. However, this solution exclusively relied on a semantic description (no axiomatisation was available) and was not clearly identified at that moment with respect to other logical approaches in the literature. Here we shall build on the results of [4] by identifying the HT^2 frames as semantical structures of a kind previously studied in logic in order to capture weak types of negation. Once we have established that partial stable models can be seen as minimal models in this semantics, the main task remaining is to axiomatise the underlying logic and prove that it is complete with respect to the frames in question. The paper reports on-going research and does not attempt to tackle all the relevant issues for the logical foundations of WFS. For reasons of space not all the technical results can be presented in full.

The paper is organised as follows. The next section introduces the logical families into which HT^2 frames have been classified. We first review Dosen semantics N, a general framework for dealing with weak negation, and then proceed to study a particular case N^* that results from combination with Routley semantics (also used in [10] for paraconsistent answer sets). Section 3 begins revisiting HT^2 frames from [4], but slightly adapted now to Dosen's definitions to show that, in fact, they are a particular case of N^* frames. We also present in this section a 6-valued characterization of HT^2 that sheds some new light on the comparison to Przymusinski's 3-valued definition of partial stable models. The main contribution of this paper is the axiomatisation of HT^2 presented in Section 3.2. Section 4 concludes the paper and outlines future work.

2 Dosen semantics for N

The logic we are going to investigate is an extension of a logic introduced by Dosen in [5] (see also [6]) which he denotes[4] by N. Dosen's aim was to study logics weaker than Johansson's minimal logic. We recall here the main definitions and facts regarding N. Formulas of N are built-up in the usual way using atoms from a given propositional

[4] We hope that the reader does not get confused by the fact that Dosen uses the same denotation N often used for Nelson's constructive logic. Both logics have the same positive fragment but *essentially different* negations.

signature At and the standard logical constants: $\wedge$, $\vee$, $\rightarrow$, $\neg$, respectively standing for conjunction, disjunction, implication and negation. We write For to stand for the set of all well-formed formulae of this language. The rules of inference for N are *modus ponens* and the rule

$$\frac{\alpha \rightarrow \beta}{\neg\beta \rightarrow \neg\alpha}$$

The set of axioms contains the axiom schemata of positive logic plus:

$$\neg\alpha \wedge \neg\beta \rightarrow \neg(\alpha \vee \beta) \tag{1}$$

Definition 1 **(N model).** *A* model *for N is a quadruple $\mathcal{M} = \langle W, \leq, R, V \rangle$ such that:*

i) *W is a non-empty set (of worlds),*
ii) *$\leq\ \subseteq W^2$ is a partial ordering among worlds,*
iii) *$R \subseteq W^2$ is an accessibility relation among worlds verifying $(\leq R)\ \subseteq\ (R \leq^{-1})$,*
iv) *and finally, V is a valuation function from $At \times W \longrightarrow \{0, 1\}$ satisfying:*

$$V(p, u) = 1\ \ \&\ \ u \leq w \ \ \Rightarrow\ \ V(p, w) = 1 \tag{2}$$

□

Sometimes, it may be technically convenient to treat a valuation V as a mapping $At \longrightarrow 2^W$ instead, where intuitively $V(p) = \{w \in W \mid V(p, w) = 1\}$.

As the reader may have already observed, the main difference with respect to intuitionistic frames is the presence of a new accessibility relation R that will be used for interpreting negation, while $\leq$ remains for implication. In this way, in order to extend valuation V on all formulas, we use for positive connectives the same conditions as in the case of intuitionistic logic, but for negation, we use instead the following condition:

$$V(\neg\varphi, w) = 1 \text{ iff for every } w' \text{such that } wRw',\ \ V(\varphi, w') = 0.$$

A proposition φ is said to be *true* in an N model $\mathcal{M} = \langle W, \leq, R, V \rangle$, if $V(\varphi, v) = 1$, for all $v \in W$. A formula φ is *valid*, in symbols $\models \varphi$, if it is true in every N model. It is easy to prove by induction that condition (2) above holds for any formula φ, ie

$$V(\varphi, u) = 1\ \ \&\ \ u \leq w \Rightarrow V(\varphi, w) = 1. \tag{3}$$

Moreover, N is complete, that is, $\models \varphi$ iff φ is a theorem of N.

2.1 Some properties of Dosen semantics

An interesting fact about the semantics of N is that one can replace the clause $(\leq R) \subseteq (R \leq^{-1})$ in Definition 1 by the alternative condition $(\leq R \leq^{-1}) \subseteq (R \leq^{-1})$ *without changing the class of models*. It is sometimes convenient to define a new relation $\widehat{R}$ as $\widehat{R} := (R \leq^{-1})$ so that, for instance, the alternative condition can be rephrased as $(\leq \widehat{R}) \subseteq \widehat{R}$. The new models using $\widehat{R}$ and this new condition are called *condensed* and satisfy the same formulas as the original N models.

As Dosen observes, by imposing additional conditions on the relations $\leq$ and R, new formulas become valid in N models. For example we have the following:

Proposition 1. *Dosen semantics satisfies the following correspondences:*

1. $\models \alpha \to \neg\neg\alpha$ *iff* $\widehat{R}$ *is symmetric.*
2. $\models \neg\neg(\alpha \to \alpha)$ *iff* $\forall u(\exists v(vRu) \Rightarrow \exists w(uRw))$.
3. $\models \neg(\alpha \to \alpha) \to \beta$ *iff* $\forall u \exists v(uRv)$. □

Completeness proofs for N (and the extension that will concern us) can be obtained via the method of canonical models. We now sketch this. First we say that a set of formulas Γ is a *theory* if it is deductively closed and a *prime* theory if it additionally satisfies the disjunction property: $\Gamma \vdash \alpha \vee \beta \Rightarrow \Gamma \vdash \alpha$ or $\Gamma \vdash \beta$. Next we note a standard lemma that for any extension S of N and any sets of formulas Σ and Δ, if $\Sigma \nvdash \Delta$ then there is a prime theory $\Gamma \supseteq \Sigma$ such that $\Gamma \nvdash \Delta$. Here $\Sigma \vdash \Delta$ means $\Sigma \vdash \psi_1 \vee \ldots \vee \psi_n$ for some $\psi_1, \ldots, \psi_n \in \Delta$. On this basis one defines canonical models as follows.

Definition 2 (Canonical model). *Let S be any extension of N. The* canonical S frame *is the triple $\langle W_c, \leq_c, R_c \rangle$ where*

1. $W_c := \{\Gamma : \Gamma$ *is a prime theory wrt* $S\}$.
2. $\Gamma \leq_c \Delta := \Gamma \subseteq \Delta$, *for* $\Gamma, \Delta \in W_c$.
3. $\Gamma R_c \Delta := \Gamma_\neg \cap \Delta = \emptyset$, *where* $\Gamma, \Delta \in W_c$ *and* $\Gamma_\neg := \{\varphi : \neg\varphi \in \Gamma\}$.

The canonical S model *is the canonical S frame together with the mapping V_c from the set of atoms to the power set of W_c such that $V_c(p) := \{\Gamma : p \in \Gamma\}$.* □

It is not hard to check that $(\leq_c R_c) \subseteq R_c$. This means that the canonical model is indeed an N model, moreover, it is a condensed N model. The main lemma needed for a completeness proof is the following.

Lemma 1. *In the canonical S model, for every $\Gamma \in W_c$ and every φ, $\Gamma \vdash \varphi \Leftrightarrow \varphi \in \Gamma$.*

The *proof* is by induction on the complexity of φ. □

The completeness property follows by noting that if $\nvdash_N \varphi$ then there is a prime theory Γ such that $\varphi \notin \Gamma$. It follows from Lemma 1 that φ does not hold in the canonical model and therefore is not N-valid. Since the canonical model is condensed we obtain that N is complete wrt the class of condensed N models.

2.2 De Morgan laws and Routley semantics

Let us consider now the logic N^* obtained by adding to N the following axioms

$$\neg(\alpha \to \alpha) \to \beta \tag{4}$$

$$\neg(\alpha \wedge \beta) \to \neg\alpha \vee \neg\beta \tag{5}$$

Thus, both De Morgan laws are provable in N^*. Moreover, taking into account Proposition 1 and axiom 4, negations of tautologies are false at every world of any N^* model, ie intuitionistic negation '$-$' is definable in N^* as: $-\alpha := \alpha \to \neg(p_0 \to p_0)$.

Definition 3 (N^* model). *A condensed N model is called an N^** model *if*

$\forall x \exists x^* (xRx^* \wedge \forall y (xRy \Rightarrow y \leq x^*)$ □

It can be easily checked that the new axioms of N^* are valid in all N^* models. The completeness of N^* wrt to the class of N^* models follows from

Lemma 2. *For an extension S of N^*, the canonical S model is an N^* model.*

Proof. For a non-trivial prime theory $\Gamma \in W_c$ the greatest theory accessible from Γ can be defined as $\Gamma^* := For \setminus \Gamma_\neg$, ie $\alpha \in \Gamma^* \;\Leftrightarrow\; \neg\alpha \notin \Gamma$. □

The fact that in N^* models for every world x there exists the greatest world x^* R-accessible from x implies that the validity of negated formulas at x is equivalent to $x \models \neg\varphi \;\;\Leftrightarrow\;\; x^* \not\models \varphi$. This observation allows defining a Routley style semantics [15] for extensions of N^*. A Routley frame is a triple $\langle W, \leq, * \rangle$, where W is a set, $\leq$ a partial order on W and $* : W \to W$ is such that $x \leq y$ iff $y^* \leq x^*$. A Routley model is a Routley frame together with a valuation $V : At \times W \longrightarrow \{0, 1\}$ like for N models.

Let S be an N^* extension. A canonical Routley S model $\langle W_c, \subseteq, *^c \rangle$ is defined similarly to canonical model for an N extension, excepting that $\Gamma^{*^c} := For \setminus \Gamma_\neg$. The completeness of N^* wrt to just defined semantics can be proved in a standard way.

3 HT^2-models

In the semantics for intermediate or superintuitionistic logics, the so-called logic of *here-and-there* can be captured by rooted Kripke frames with two elements, commonly denoted by h and t and called 'here' and 'there', with $h \leq t$. As shown in Pearce [11] this logic can be used as a foundation for the stable model semantics for logic programs. In Cabalar [4] a notion of HT^2 model was introduced and studied in order to capture partial stable models for logic programs. The motivation for the notation is that HT^2 models are based on frames that include for each world w in an HT-model an additional world w' accessible from w via the $\leq$ relation. In addition, just as we have $h \leq t$ in an HT-model, we have also $h' \leq t'$ in an HT^2-model. More precisely we define HT^2 in terms of N models as follows.

Definition 4 (HT^2 model). *An HT^2 model is an N model $\mathcal{M} = \langle W, \leq, R, V \rangle$ such that (i) W comprises 4 worlds denoted by h, h', t, t', (ii) $\leq$ is a partial ordering on W satisfying $h \leq t$, $h \leq h'$, $h' \leq t'$ and $t \leq t'$, (iii) $R \subseteq W^2$ is given by hRh', $h'Rh$, tRt', $t'Rt$, hRt', $h'Rt$. (iv) V is an N-valuation.* □

An informal intuition for this structure from the logic programming perspective would be the following. In HT, the valuation for world t, call it I, plays the role of an interpretation we could call the "initial assumption" used to rule out all default negations in the program Π leading to the well-known program reduct [7] Π^I. Valuation for world h, write it J, would correspond to each classical model of Π^I. For partial stable models, as defined in [12], both interpretations I and J become three-valued. This is captured in HT^2 by dealing with (h, h') and (t, t'), so that I makes true those atoms in t, false those not in t' and undefined all the rest, and the same holds for J wrt (h, h').

For clarity sake, relations $\leq$ and R are depicted in Figure 1. An interesting observation that can be checked in the figure is that when we force $h = h'$ and $t = t'$ we actually obtain that $\leq$ and R collapse into the same relation and, in fact, the whole structure becomes an HT frame. Thus, it is easy to see that:

Proposition 2. *Any valid formula in HT^2 is also a valid formula in HT.* $\square$

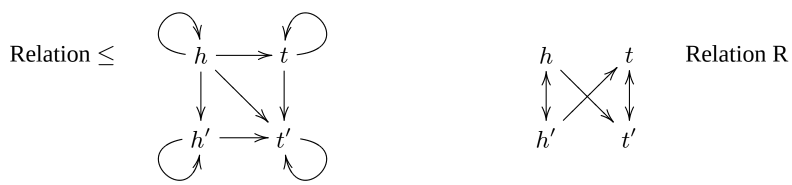


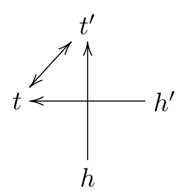
Fig. 1. Accessibility relations for HT^2 models.

Another important fact that can be easily checked is that $\leq, R$ satisfy the conditions for N-models. Moreover, we have:

Proposition 3. *In HT^2 models the following formulas are valid:*

$$\alpha \rightarrow \neg\neg\alpha, \;\; \neg(\alpha \rightarrow \alpha) \rightarrow \beta, \;\; \neg(\alpha \wedge \beta) \rightarrow \neg\alpha \vee \neg\beta. \qquad \square$$

According to Proposition 3 the HT^2 frame defines an extension of N^* and we can replace the above defined HT^2 models by models based on the Routley frame $\mathcal{W}^{HT^2} = \langle W^{HT^2}, \leq, * \rangle$, where $W^{HT^2} = \{h, h', t, t'\}$ and the ordering $\leq$ and the action of $*$ are presented at the following diagram.



That is, $h^* = t^* = t'$ and $(h')^* = (t')^* = t$. Now, fix some HT^2 model $\mathcal{M} = \langle \mathcal{W}^{HT^2}, V \rangle$. For $w \in W^{HT^2}$ let us denote
$\Delta_w := \{\varphi : w \models \varphi\}$.

Lemma 3. *For an arbitrary HT^2 model $\mathcal{M} = \langle \mathcal{W}^{HT^2}, V \rangle$ the following holds.*

1. *Δ_w is a prime HT^2 theory for any $w \in W^{HT^2}$.*
2. *$\Delta_u \subseteq \Delta_v$ iff $u \leq v$.*
3. *$\Delta_{t'} = \Delta_h^*$ and $\Delta_t = \Delta_{h'}^*$.*
4. *$\varphi \rightarrow \psi \in \Delta_w$ iff for all $v \geq w$ either $\varphi \notin \Delta_v$ or $\psi \in \Delta_v$.* $\square$

The proof follows from the definition of validity of formulas in Routley models. Due to this lemma Routley HT^2 models can be identified with quadruples of prime HT^2 theories $\{\Delta_h, \Delta_{h'}, \Delta_t, \Delta_{t'}\}$ satisfying all conditions of Lemma 3. Given a quadruple $\{\Delta_h, \Delta_{h'}, \Delta_t, \Delta_{t'}\}$ a valuation function can be reconstructed as follows: $V(p, w) = 1$ iff $p \in \Delta_w$.

Proposition 4. *If* $\varphi \notin HT^2$*, then* $\nvdash_{HT^2} \{\varphi\} \cup \{\psi \wedge \neg\psi : \psi \in For\}$. □

Corollary 1. HT^2 *is closed under the rule* $\frac{\alpha \vee (\beta \wedge \neg\beta)}{\alpha}$. □

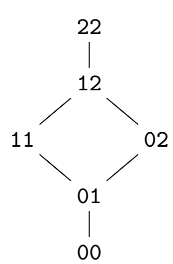
An alternative way of describing HT^2 semantics is by defining its logical matrix, in a similar way as happens with HT and the corresponding Gödel's three-valued logic G_3. In HT, the possible "situations" of a formula φ with respect to an interpretation are three, and can be labelled as $\mathtt{0} := \emptyset,\ \mathtt{1} := \{t\},\ \mathtt{2} := \{h, t\}$ (respectively called *false*, *undefined* and *true*) where each set shows the worlds at which the formula is satisfied. The tables for G_3 are then derived from the description of satisfaction under HT Kripke frames. In the case of an HT^2 frame, it is easy to see that the number of possible situations for any formula φ rises to six. For comparison purposes, we label each value as a pair xy with x and y varying from $\{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$ so that x would correspond to G_3 value for worlds (h, h') and y to G_3 value for (t, t'). As a result, we have:

$$\mathtt{00} := \emptyset,\ \mathtt{01} := \{t'\},\ \mathtt{11} := \{h', t'\},\ \mathtt{02} := \{t, t'\},\ \mathtt{12} := \{h', t, t'\},\ \mathtt{22} := W^{HT^2}.$$

becoming an algebra of 6 cones: $\mathcal{A}^{HT^2} := \langle\{\mathtt{00}, \mathtt{01}, \mathtt{11}, \mathtt{02}, \mathtt{12}, \mathtt{22}\}, \vee, \wedge, \rightarrow, \neg\rangle$ where $\vee$ and $\wedge$ are set theoretical join and meet, whereas $\rightarrow$ and $\neg$ are defined as follows:

$$x \rightarrow y := \{w : w \leq w' \Rightarrow (w' \in x \Rightarrow w' \in y)\}, \quad \neg\, x := \{w : w^* \notin x\}.$$

The only distinguished element is $\mathtt{22}$. The lattice structure of this algebra can be described by the condition $xy \leq zt \Leftrightarrow x \leq z\ \&\ y \leq t$ and is shown in Figure 2, together with the resulting truth-tables.



φ	$\neg\varphi$
00	22
01	11
11	11
02	00
12	00
22	00

$\rightarrow$	00	01	11	02	12	22
00	22	22	22	22	22	22
01	00	22	22	22	22	22
11	00	02	22	02	22	22
02	00	11	11	22	22	22
12	00	01	11	02	22	22
22	00	01	11	02	12	22

Given $V(\phi) = xy$ and $V(\psi) = zt$:

$$V(\phi \wedge \psi) = uv \Leftrightarrow u = min(x, z)\ \&\ v = min(y, t)$$
$$V(\phi \vee \psi) = uv \Leftrightarrow u = max(x, z)\ \&\ v = max(y, t)$$

Fig. 2. Lattice structure and truth tables for the 6-valued HT^2 description.

3.1 Minimal HT^2 models and partial stable models

Consider an HT^2 model $\mathcal{M} = \langle W, \leq, ^*, V\rangle$. More succinctly, let us represent $\mathcal{M}$ as a pair of ordered pairs $(\langle H, H'\rangle, \langle T, T'\rangle)$, where the upper-case letters denote the sets of atoms verified at the corresponding point or world. Even more succinctly we can denote an unprimed, primed pair in the form $\mathbf{H}$ and $\mathbf{T}$, so $\mathcal{M}$ is represented simply as $(\mathbf{H}, \mathbf{T})$, with $\mathbf{H} = \langle H, H'\rangle$ and $\mathbf{T} = \langle T, T'\rangle$. Such pairs can be partially ordered as follows. We say in general that $\mathbf{H} \leq \mathbf{T}$ if $H \subseteq T$ and $H' \subseteq T'$. Notice that by the semantics, if $(\mathbf{H}, \mathbf{T})$ is a model then necessarily $\mathbf{H} \leq \mathbf{T}$. Moreover, for any theory Π note that if $(\mathbf{H}, \mathbf{T}) \models \Pi$ then also $(\mathbf{T}, \mathbf{T}) \models \Pi$.

The ordering $\leq$ can be extended to a partial ordering $\trianglelefteq$ among models as follows. We set $(\mathbf{H}_1, \mathbf{T}_1) \trianglelefteq (\mathbf{H}_2, \mathbf{T}_2)$ if (i) $\mathbf{T}_1 = \mathbf{T}_2$; (ii) $\mathbf{H}_1 \leq \mathbf{H}_2$. A model $(\mathbf{H}, \mathbf{T})$ in which $\mathbf{H} = \mathbf{T}$ is said to be *total*. Note that the term *total* model does not refer to the absence of undefined atoms (i.e., a total model need not be *complete*).

We are interested here in a special kind of minimal model that we call a partial equilibrium model. Let Π be a theory.

Definition 5 (Partial equilibrium model). *A model $\mathcal{M}$ of Π is said to be a* partial equilibrium *model of Π if (i) $\mathcal{M}$ is total; (ii) $\mathcal{M}$ is minimal among models of Π under the ordering $\trianglelefteq$.* □

In other words a partial equilibrium model of Π has the form $(\mathbf{T}, \mathbf{T})$ and is such that if $(\mathbf{H}, \mathbf{T})$ is any model of Π with $\mathbf{H} \leq \mathbf{T}$, then $\mathbf{H} = \mathbf{T}$.

Among partial equilibrium models of a theory we can define an additional ordering $\preceq$ as follows. We set $(\mathbf{T}_1, \mathbf{T}_1) \preceq (\mathbf{T}_2, \mathbf{T}_2)$ if (i) $T_1 \subseteq T_2$; (ii) $T_2' \subseteq T_1'$.

Definition 6 (Well-founded model). *A partial equilibrium model of a theory Γ that is $\preceq$-minimal among all the partial equilibrium models of Γ is called a (HT^2)* well-founded model *of Γ.* □

Theorem 1 (Shown in Theorem 3 of [4]). *Partial equilibrium models (resp. HT^2 well-founded model) for normal logic programs coincide with partial stable models (resp. well-founded model).* □

3.2 The axioms of HT^2

Although HT^2 was originally described as a class of frames, it can be considered a logic in the sense that it defines a set of formulas: the ones that are valid. We will see next that, in fact, a more constructive (and perhaps more standard) definition of HT^2 is also possible using a calculus description, that is, a set of axioms and inference rules.

Let HT^* be an N^* extension obtained by adding the following axioms:

A1. $-\alpha \vee - - \alpha$
A2. $-\alpha \vee (\alpha \rightarrow (\beta \vee (\beta \rightarrow (\gamma \vee -\gamma))))$
A3. $\bigwedge_{i=0}^{2}((\alpha_i \rightarrow \bigvee_{j \neq i} \alpha_j) \rightarrow \bigvee_{j \neq i} \alpha_j) \rightarrow \bigvee_{i=0}^{2} \alpha_i$
A4. $\alpha \rightarrow \neg\neg\alpha$
A5. $\alpha \wedge \neg\alpha \rightarrow \neg\beta \vee \neg\neg\beta$
A6. $\neg\alpha \wedge \neg(\alpha \rightarrow \beta) \rightarrow \neg\neg\alpha$

A7. $\neg\neg\alpha \vee \neg\neg\beta \vee \neg(\alpha \rightarrow \beta) \vee \neg\neg(\alpha \rightarrow \beta)$
A8. $\neg\neg\alpha \wedge \neg\neg\beta \wedge (\beta \rightarrow \alpha) \rightarrow \alpha \vee (\alpha \rightarrow (\beta \vee -\beta))$

and the rule $\frac{\alpha \vee (\beta \wedge \neg \beta)}{\alpha}$.

Proposition 5. *The frame $\mathcal{W}^{HT^*}$ of the canonical HT^* model satisfies the following properties: (i) $\mathcal{W}^{HT^*}$ is strongly directed; (ii) $\mathcal{W}^{HT^*}$ is of depth 3; and (iii) each element of $\mathcal{W}^{HT^*}$ has at most two immediate successors.*

Proof. Items of this proposition can be inferred from axioms A1, A2 and A3 respectively in the same way as for superintuitionistic logics determined by these axioms. □

Theorem 2. $HT^* = HT^2$.

Proof. Although we have not space enough here to include a formal proof, the result follows from the properties of the canonical HT^* model listed in the previous proposition and a detailed analysis of prime HT^*-theories. □

4 Related work and conclusions

It is very natural to want to remove some of the syntactic restrictions on normal logic programs under well-founded semantics and even to generalise well-founded inference to the full language of propositional or predicate logic and compare it to other nonmonotonic logical systems. There have been numerous attempts in this direction but very little consensus reached on which extensions are the natural or appropriate ones. The main difficulty is that there are different views about which features of well-founded inference are the essential ones and need to be preserved in any extension or generalisation. Our own view can be summarised as follows. In order to build a general system of nonmonotonic logic based on well-founded inference we need to identify an underlying monotonic logical framework to be used as a basis. The natural choice is a logic in which well-foundedness or partial stability can be expressed as a simple minimality condition. The condition of equilibrium that captures stable models in the logic of here-and-there can be readily generalised to a minimality condition that captures partial stability in a logic HT^2 that corresponds in a natural way to HT. In this paper we have shown how the resulting logic has a six-valued truth matrix and can be axiomatised as an extension of Dosen's logic N. Although the negation of HT^2, corresponding to the well-founded negation, is rather weak, intuitionistic negation is actually definable in HT^2.

Previous attempts to provide a more logical characterisation of well-founded semantics have included eg [2,3] which focus more on Gentzen style deduction rather than model-theoretic minimality conditions and [14] which proposes an infinite valued logic not easily recognisable among normal many-valued logics. [1] develops an approach to WFS and its extensions using semantical frames, but these are mainly generalisations of those of HT^2 described in [4] and no logical axiomatisation of the semantics is attempted.

An approach to be mentioned apart is the modal characterisation presented in [13] called *Autoepistemic Logic of Beliefs*. This approach also allows one to describe well-founded models for disjunctive programs and, in principle, a full nesting of logical operators, although it has not yet been classified into any established logical family and does not seem to display a similar property to Proposition 2 in relation to here-and-there. However, a comparison with this framework is planed for future work.

The present paper describes work in progress that will continue to examine many more issues in the foundations of well-founded semantics. They include the study of (i) metalogical and computational properties of WF inference in the case of disjunctive logic programs or even general propositional theories; (ii) strong equivalence [9] for programs and theories under WFS in relation to the logic HT^2; and (iii) the relation of HT^2 to well-known extensions of WFS such as WFSX with explicit negation.

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