Chapter 5. Equilibrium Logic

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February 8, 2011
Outline

1. Extending the syntax: logical interpretation
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Extending the syntax

- **Disjunctive programs**: bodies $B$ as before, but heads allow disjunctions of atoms:

  $$p_1 \lor \cdots \lor p_n \leftarrow B$$

- The reduct is defined as before, but note that $P^l$ does not have now a least Herbrand model: only minimal ones. Example:

  $$p \lor q \leftarrow t, \lnot s \quad t \leftarrow \lnot q$$

  Given $I = \{p, t\}$, $P^l$ is the program:

  $$p \lor q \leftarrow t \quad t \leftarrow$$

  whose minimal models are $\{p, t\}$ (stable) and $\{q, t\}$ (non-stable).
The definition is adapted accordingly

Definition (stable model)

$I$ is a stable model of a disjunctive program $P$ if it is a minimal model of $P^I$.

Finding a stable model of a disjunctive program is slightly more complex: $\Sigma_2^P$-complete.

Tools for disjunctive ASP: DLV, GnT, cmodels.
Extending the syntax: logical interpretation

Adding **default negation in the head** [Inoue & Sakama 98]. Rules

\[ H \leftarrow B \]

1. **Body** \( B \) = conjunction of literals (as before). We define:

\[
B = q_1 \land \cdots \land q_n \land \not q_{n+1} \land \cdots \land \not q_m
\]

2. **Head** \( H \) = disjunction of literals. We define:

\[
H = p_1 \lor \cdots \lor p_k \lor \not p_{k+1} \lor \cdots \lor \not p_s
\]
We adapt the definition of reduct as follows:

$$P^I = \{ H^+ \leftarrow B^+ \mid I \models B^- \land \neg H^- \}$$

Example: given $I = \{a, d, c\}$ and program

$$b \lor \neg a \lor \neg d \leftarrow d, \neg e, \neg h$$

$B^- = \neg e \land \neg h$ and $\neg H^- = \neg(\neg a \lor \neg d) = (a \land d)$. As $I \models B^- \land \neg H^-$, its reduct would correspond to:

$$b \leftarrow h$$
Extending the syntax

Stable models are defined as before: \( I \) minimal model of \( P^l \).

Example: \( P \) is the program

\[
p \lor \neg p
\]

\[
q \leftarrow \neg p
\]

\[
\begin{array}{c|c|c}
I & P^l & \text{minimal models} \\
\hline
\emptyset & q & \{q\} \neq I \text{ not stable} \\
\{p\} & p & \{p\} \neq I \text{ stable!} \\
\{q\} & q & \{q\} \text{ stable!} \\
\{p, q\} & p & \{p\} \text{ not stable}
\end{array}
\]
Extending the syntax

- **Nested expressions** [Lifschitz, Tang, Turner 99]:
  
  \[ H \text{ and } B \text{ can be any combination of atoms with } \bot, \top, \land, \lor, \text{not}. \]

- Several **transformation rules** (we’ll see later) allow **reducing** nested expressions to disjunctive programs with negation in the head.

- An example: the nested rule

  \[
  a \lor \neg (b \land \neg c) \leftrightarrow d \lor \neg e
  \]

  becomes the program:

  \[
  a \lor \neg b \leftrightarrow d \land \neg c \\
  a \lor \neg b \leftrightarrow \neg e \land \neg c
  \]

- But, which is the semantics for \( \neg (a \leftarrow b) \) or \( a \leftarrow (b \leftarrow c) \)?

Let us write rules like $p \leftarrow q, \neg r$ in standard logical notation
$q \land \neg r \rightarrow p$

Equilibrium Logic [Pearce96]: generalises Answer Sets for arbitrary propositional theories.

Consists of:

1. A non-classical monotonic (intermediate) logic called Here-and-There (HT)

2. A selection of (certain) minimal models that yields nonmonotonicity
Here-and-There

- **Interpretation** = pairs \( \langle H, T \rangle \) of sets of atoms \( H \subseteq T \)

- **Intuition**: \( H \) = true atoms, \( T \) = non-false. When \( H = T \) we have a classical model.

- **Satisfaction** of formulas
  - \( \langle H, T \rangle \models p \) if \( p \in H \) (for any atom \( p \))
  - \( \land, \lor \) as always
  - \( \langle H, T \rangle \models \varphi \rightarrow \psi \) if both
    - \( \langle H, T \rangle \models \varphi \) implies \( \langle H, T \rangle \models \psi \)
    - \( \langle T, T \rangle \models \varphi \) implies \( \langle T, T \rangle \models \psi \)
      - This is the same than \( T \models \varphi \rightarrow \psi \) in classical logic.

- **Negation** \( \neg F \) is defined as \( F \rightarrow \bot \)
Here-and-There

Some properties

- $\langle T, T \rangle \models \Gamma$ is the same than $T \models \Gamma$ in classical logic.
- $\langle H, T \rangle \models \Gamma$ implies $T \models \Gamma$.
- $\langle H, T \rangle \models \neg \varphi$ iff $T \not\models \varphi$ in classical logic.
Possible alternative description using 3-valued semantics (Gödel’s logic $G_3$).

Given $M = \langle H, T \rangle$, we can define a 3-valued mapping $M : Atoms \mapsto \{0, 1, 2\}$ reading:

- $2 = (p \in H)$ = true
- $0 = (p \notin T)$ = false
- $1 = (p \in T \setminus H)$ = undefined

$\land$ returns minimum value, $\lor$ returns maximum and $M(\phi \rightarrow \psi) = 2$ if $M(\phi) \leq M(\psi)$ or returns $M(\psi)$ otherwise.
Definition (Equilibrium model)

$\langle T, T \rangle$ is an equilibrium model of a theory $\Gamma$ if:

$\langle T, T \rangle \models \Gamma$, and there is no $H \subset T$ such that $\langle H, T \rangle \models \Gamma$. 
Equilibrium Logic

- Logical techniques available: e.g., methods from many-valued semantics (tableaux, signed logics, . . .)
- Captures all previous syntax extensions, plus other non-propositional constructions:
  - **weight constraints** can be represented as nested expressions \([\text{Ferraris, Lifschitz 2005}]\);
  - **aggregates** represented by rules with **embedded implications** \([\text{Ferraris 2004}]\).
  - **ordered disjunction** from \([\text{Brewka et al 2004}]\) (LPOD) can also be captured \([\text{Cabalar 2010}]\).
Equilibrium logic

Other interesting features

- In nonmonotonic reasoning, we talk about strong equivalence of $\Gamma_1, \Gamma_2$ when, for any $\Pi$:
  $\Gamma_1 \cup \Pi$ and $\Gamma_2 \cup \Pi$ have the same (selected) models.

- $\Gamma_1, \Gamma_2$ are strongly equivalent iff they are equivalent in HT [Lifschitz et al 2001].
Equilibrium logic

Other interesting features

- Disjunctive programs with negation in the head are a (conjunctive) normal form (CNF) for Equilibrium Logic. [Cabalar & Ferraris 2007].

Theorem

The number of different logic programs (modulo strong equivalence) that can be built for a finite signature of $n$ atoms is:

$$\prod_{i=0}^{n} (2^{2i-1} + 1)^{n}$$

With $n = 2$ we get 162, with $n = 3$ around 5 million.

- Transformations into this CNF [Cabalar, Pearce & Valverde 2005].
Other interesting features

- Equilibrium Logic also covers full First Order Theories with equality [Pearce & Valverde 2004].
- Introduction of partial functions [Cabalar 2008].
- Linear temporal equilibrium logic [Cabalar & Pérez 2007].