On fuzzy closure systems for fuzzy consequence operators

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4 November, 2022

Let me paint you a situation

A woman is mugged on the street, she is in the police station giving a declaration.

Statement

"It was a furious, very tall and very strong man, I remember his piercing green eyes."

What she describes

The real mugger

Thus, fuzzy logic

We need a mathematical structure capable of working with

- Uncertainty
- Vagueness
- Incompleteness
- Errors

Fuzzy logic gives a framework where working with this kind of data is within reach.

(Fuzzy) closure operators

Definition ((Classical) closure operator)

Let $(A, \leq)$ be a partially ordered set. A mapping $\mathrm{c}\colon A \to A$ is said to be a (classical) closure operator if it satisfies the following:

1. $x \leq \mathrm{c}(x)$, for all $x \in A$
2. if $x \leq y$ then $\mathrm{c}(x) \leq \mathrm{c}(y)$, for all $x, y \in A$
3. $\mathrm{c}(\mathrm{c}(x)) = \mathrm{c}(x)$, for all $x \in A$.

Definition (Closure operator)

Let (A, ρ) be a fuzzy partially ordered set. A mapping $\mathrm{c}\colon A \to A$ is said to be a (fuzzy) closure operator if it satisfies the following:

1. $\rho(x, \mathrm{c}(x)) = 1$, for all $x \in A$
2. $\rho(x, y) \leq \rho(\mathrm{c}(x), \mathrm{c}(y))$, for all $x, y \in A$
3. $\rho(\mathrm{c}(\mathrm{c}(x)), \mathrm{c}(x)) = 1$, for all $x \in A$.

Fuzzy closure systems

Intuition

We would like closure systems in the fuzzy framework to extend the meet-subsemilattice property of closure systems in the classical case.

Definition

A crisp set $\mathcal{F} \subseteq \boldsymbol{A}$ is said to be a closure system if $\bigsqcap \boldsymbol{X} \in \mathcal{F}$, for all $\boldsymbol{X} \in \boldsymbol{L}^{\mathcal{F}}$.

Theorem

There is a one-to-one connection between closure systems and closure operators.

Fuzzy closure systems

First idea

Fuzzy closure systems would be a fuzzy sets such that the membership of an element is exactly the degree to which an element is closed.

Success

The previous idea was successful and we could formalize a (somewhat technical) definition of fuzzy closure systems.

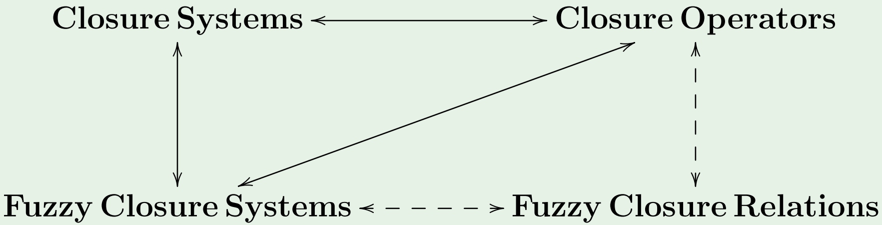
Theorem

There is a one-to-one connection between fuzzy closure systems and closure operators.

Fuzzy closure relations

Intuition

Once the closure systems have been extended to fuzzy sets, we now focus on extending the closure operators to closure relations in order to complete a diagram similar to the following one.



Fuzzy closure relations

Intuition

We would like a fuzzy closure relation $\kappa(a, b)$ to represent how close is b of being the closure of a, that is, how similar are $c(a)$ and b.

Results

Again, following this idea we find a (even more technical) definition of what a fuzzy closure relation should be. This definition turned out to be equivalent to some other approaches in the literature, such as perfect fuzzy functions or extensional hulls.

Theorem

There is a one-to-one relation between fuzzy closure systems and strong fuzzy closure relations.

Pseudointents

In the classical case...

There is a particular kind of elements called pseudo-closed elements, or pseudointents, that satisfy the following. The set of attribute-implications

$$\{p \to (p) \mid p \text{ is a pseudointent}\}$$

is a minimal basis of attribute-implications called Duquenne-Guigues basis or stem basis.

Work in progress

The extension of this concept to the fuzzy framework depends strongly on the fuzzy closure structures defined in the slides. The search for an appropriate definition of pseudointent and the study of minimality of the basis they define is our current project.

Conclusions and further work

Conclusions

- We have done a lot of technical work to broaden the amount of tools available for applications.

Further work

- Use these recent results to face the problem that initially generated all this line of research in the first place, that is, the fuzzy extension of pseudo-intents. (Minimal basis)
- A study on fuzzy consequence operators can be done using the fuzzy closure structures presented here.

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