New insights on the intuitionistic interpretation of Default Logic

Pedro Cabalar and **David Lorenzo**[1]

Abstract. In this work we further investigate the relation, first found by Truszczyński, between modal logic S4F and Default Logic (DL), analyzing some interesting properties and showing its application to other general non-monotonic formalisms. For comparison purposes, we start defining a subset of S4F we called *Intuitionistic Default Logic* (IDL), which consists in incorporating an additional set of propositional operators. These operators are translated into modal formulas using Gödel's pattern for encoding Intuitionistic Logic into S4. Then, we prove that, under a particular models minimization policy, IDL generalizes Turner's *Nested Default Logic* (NDL), in the sense that the latter does not allow nesting or combining the rule conditional operator. This result is also used to show that strong equivalence of default theories can be reduced to a simple S4F equivalence test. Finally, we prove that IDL also generalizes Pearce's *Equilibrium Logic*, which encodes logic programs into the intermediate logic of Here-and-There.

1 INTRODUCTION

The advantages of capturing a Nonmonotonic Reasoning (NMR) formalism in logical terms are both theoretical and practical. From a theoretical point of view, we get a clear semantic interpretation for all the constructs of the NMR formalism and, usually, a glimpse on their possible generalization. From a practical point of view, we can convert the study of NMR theories into theorem proving inside the underlying logical framework. While some NMR formalisms [14] are defined in logical terms from the very beginning, encodings for other popular nonmonotonic frameworks have been frequently considered in the literature. For instance, in the case of Reiter's Default Logic (DL) [19], there exists a whole family of nonmonotonic modal characterizations studied in [13]. Another example, directly related to DL, is the recently renewed interest in logical characterizations of Programming (LP) under *stable models* semantics [4]. The most relevant encoding in this case is, perhaps, Pearce's *Equilibrium Logic* [17] which relies on Heyting's intermediate logic of *Here-and-There* (HT).

As shown in [10], the HT characterization has an important additional advantage: strong equivalence of logic programs corresponds to HT-equivalence of their logical translations. We say that two logic programs P_1 and P_2 are *strongly equivalent* when $P_1 \cup P$ and $P_2 \cup P$ yield the same consequences, for any additional set of rules P. Usually, a logical encoding provides a sufficient condition for strong equivalence, but perhaps not a necessary one – our claim is that this last feature is a *desirable property.* Apart from HT, other approaches have been used to provide necessary and sufficient conditions for strong equivalence of logic programs (for instance, relying on classical propositional logic [18, 12], three-valued logic [1] or a whole family of intermediate logics [3]). In all these cases, the result is still applicable for the general syntax of *nested* LP [11], where default negation, conjunction and disjunction can be freely combined both in the head and the body of rules. Strong equivalence for default theories has been characterized by Turner in [21], who actually deals with a generalization of DL called *Nested Default Logic* (NDL) – essentially, nested LP where atoms can be replaced by classical formulas.

In this paper, we study some interesting properties of the translation of DL into (non-monotonic) modal logic S4F. This translation was introduced by Truszczyński in [20] and was later observed to follow Gödel's general pattern [6] for encoding[2] Intuitionistic Logic into S4. In our work, the straightforward application of Gödel's pattern allows us to propose one more generalization of DL we have called *Intuitionistic Default Logic* (IDL), and which we show extremely useful for comparison purposes. This generalization consists in freely combining intuitionistic operators, but allowing classical formulas to play the role of "atoms." As a result, we can prove that: (1) IDL generalizes NDL, in the sense that it does not impose any restriction for combining intuitionistic operators (curiously, despite of its name, NDL does not allow nesting default rules); (2) equivalence of IDL theories (under S4F) is a necessary and sufficient condition for strong equivalence of default theories; and (3), IDL is a proper generalization of HT, when we consider classical formulas instead of atoms.

The rest of the paper is organized as follows. Section 2 contains some definitions and notation, together with a brief recall of modal logic. The next section introduces a nonmonotonic version of S4F and proceeds then to define IDL inside this framework. Sections 4 and 5 study the relation to NDL and HT, respectively. Finally, Section 6 concludes the paper. The proofs for the main results have been included in [2].

2 PRELIMINARIES

All languages described in the paper are assumed to be propositional. The finite set At, called *signature*, contains all the

[1] Dept. of Computer Science, University of Corunna, Spain. email: {cabalar,lorenzo}@dc.fi.udc.es

[2] Gödel's pattern has been frequently used for discovering modal companions of intermediate logics. Its application to nonmonotonic formalisms is first explicitly mentioned in [16].

propositional atoms. We adopt the following notation for languages $[op_1 \ op_2 \ \dots \ op_n](At)$ standing for all the formulas constructed with elements of signature At combined with operators op_i. Note that signature can be different from At: some languages can use as "atoms" the set of formulas of another language. Operators $[\vee \ \wedge \ \supset \equiv \ \neg \perp \top]$ will be called *classical* and defined with their standard arity and precedence. We will also use a second set of operators $[not \ , \ ; \ \leftarrow]$ called *intuitionistic* and inherited from LP syntax with the usual arity and precedence (',' stands for conjunction and ';' for disjunction). The reason for the name "intuitionistic" is that, in fact, we will use the same notation for logic programs, default theories and intuitionistic logic, in order to reduce translation efforts. When combined, we assume that classical operators have higher priority than intuitionistic ones.

The set of *classical formulas*, $\mathcal{L}$, is defined as $[\vee \ \neg \ \perp](At)$. (the rest of classical operators are derived from the previous ones in the usual way). We will use capital letters $F, G, \dots$ to denote classical formulas.

Given a set of atoms At, a *propositional interpretation*, I, is any subset $I \subseteq At$ containing all the atoms valuated as *true*. If S is a set of propositional interpretations, then we define $Th(S)$ as the set of classical formulas that are satisfied by all interpretations in S:

$$Th(S) \stackrel{\text{def}}{=} \{F \in \mathcal{L} \mid \text{for all } I \in S, I \models F\}$$

For the following definitions regarding modal logic, we have mostly followed [13]. The language of *modal formulas* is defined as $\mathcal{L}_L = [\vee \ \neg \ \perp \ L](At)$, that is, classical operators plus an additional unary operator L, called the *necessity* functor. The dual operator M, called *possibility*, is derived from L as $M\phi \stackrel{\text{def}}{=} \neg L \neg \phi$. A *modal theory* is any subset of $\mathcal{L}_L$.

A *modal logic* $\mathcal{S}$ is usually described in terms of a set of axioms. We write $T \vdash_{\mathcal{S}} \phi$ to express that formula ϕ is *derivable* from T and axioms of $\mathcal{S}$ using the inference rules of *modus ponens* (MP) and *necessitation* (N):

$$\frac{\phi, \phi \supset \psi}{\psi} \quad \text{(MP)} \qquad\qquad \frac{\vdash_S \phi}{\vdash_S L\phi} \quad \text{(N)}$$

We define the *consequences* of any modal theory T under logic $\mathcal{S}$ as $Cn_{\mathcal{S}}(T) \stackrel{\text{def}}{=} \{\phi \in \mathcal{L}_L \mid T \vdash_{\mathcal{S}} \phi\}$.

We will be particularly interested in the following set of axioms:

- **k**. $L(\phi \supset \psi) \supset (L\phi \supset L\psi)$
- **t**. $L\phi \supset \phi$
- **4**. $L\phi \supset LL\phi$
- **f**. $\phi \wedge ML\psi \supset L(M\phi \vee \psi)$
- **5**. $M\phi \supset LM\phi$

Modal logic S4 corresponds to the set of axioms $\{\mathbf{k}, \mathbf{t}, \mathbf{4}\}$, logic S4F is defined as S4+$\{\mathbf{f}\}$ whereas S5 corresponds to S4+$\{\mathbf{5}\}$.

The semantics of these three logics can be captured in terms of the so-called Kripke models. A *Kripke model* is a triple $\mathcal{M} = \langle W, R, V \rangle$ where W is a nonempty set (whose elements are the *worlds* of $\mathcal{M}$), $R \subseteq W \times W$ is called the *accessibility relation* among worlds, and finally, V is a set of propositional interpretations, one I_w for each world $w \in W$. We define when $\mathcal{M}$ *satisfies* a modal formula ϕ at a given world w, written $(\mathcal{M}, w) \models \phi$, recursively as follows:

1. $(\mathcal{M}, w) \models p$ iff $p \in I_w$ for any atom p.
2. $(\mathcal{M}, w) \models \neg\phi$ iff $(\mathcal{M}, w) \not\models \phi$.
3. $(\mathcal{M}, w) \models \phi \vee \psi$ iff $(\mathcal{M}, w) \models \phi$ or $(\mathcal{M}, w) \models \psi$.
4. $(\mathcal{M}, w) \models L\phi$ iff for all $w' \in W$ s.t. wRw', $(\mathcal{M}, w') \models \phi$.

When ϕ is satisfied at any world w of $\mathcal{M}$ we simply write $\mathcal{M} \models \phi$ and say that ϕ is *valid* in $\mathcal{M}$. It is not difficult to see that:

Proposition 1 *If relation R is reflexive:*
$\mathcal{M} \models \phi$ *iff* $\mathcal{M} \models L\phi$. □

Similarly, a modal theory T is *valid* in $\mathcal{M}$, also written $\mathcal{M} \models T$, when all the formulas in T are valid in $\mathcal{M}$. Given a class of Kripke models $\mathcal{K}$, any theory T and any formula ϕ, we write $T \models_{\mathcal{K}} \phi$ to represent that any $\mathcal{M} \in \mathcal{K}$ such that $\mathcal{M} \models T$ satisfies $\mathcal{M} \models \phi$. As expected, $\models_{\mathcal{K}} \phi$ means that ϕ is true in any Kripke model of class $\mathcal{K}$.

A modal logic $\mathcal{S}$ is said to be *characterized* by a class of Kripke models $\mathcal{K}$ iff deduction and entailment coincide, that is, for any T and ϕ: $T \vdash_{\mathcal{S}} \phi$ iff $T \models_{\mathcal{K}} \phi$.

The class of Kripke models characterizing S4 consists of those with a transitive and reflexive accessibility relation. Kripke models for S5 have the shape $\langle W, W \times W, V \rangle$ (that is, they are transitive, reflexive and symmetric). Usually, S5-models are directly represented as $\langle W, V \rangle$. Finally, the most interesting structure for our purpose is the class of Kripke models characterizing S4F, which have the shape $\langle W, (W_1 \times W) \cup (W \times W_2), V \rangle$ where $W = W_1 \cup W_2$, $W_1 \cap W_2 = \emptyset$ and $W_2 \neq \emptyset$. In other words, each S4F-model consists of a pair of S5 clusters, W_1 and W_2, where W_1 is fully connected to W_2. We will directly represent the S4F-model as $\langle W_1, W_2, V \rangle$. Note that, when $W_1 = \emptyset$, we can consider that it actually amounts to an S5-model $\langle W_2, V \rangle$.

It is perhaps interesting to note that the number of different modalities in each of the three mentioned logics is relatively small. By *different modality* we mean a string of modal operators which cannot be equivalently reduced into a smaller string. It is well-known [8] that S4 has the following six different modalities[3] L, M, LM, ML, LML, MLM. For instance, in S4 we have:

$$LL\phi \equiv L\phi \qquad (1)$$
$$MM\phi \equiv M\phi \qquad (2)$$

In S5, there are only two modalities: L and M. In a similar way, the following theorem:

$$ML\phi \supset LM\phi \qquad (3)$$

from S4F can be used to prove:

$$LML\phi \equiv ML\phi \qquad (4)$$
$$MLM\phi \equiv LM\phi \qquad (5)$$

showing that this logic has the four modalities L, M, LM and ML. In fact, we can just consider L and ML, seeing M and LM as their respective negations. The following theorems of S4F describe some unfolding properties of these modalities which will be especially useful later:

[3] We omit everywhere the case of non-modal formulas, which could also be considered as an additional "empty" modality.

$$L\neg ML\phi \equiv \neg ML\phi \quad (6)$$
$$L(\phi \wedge \psi) \equiv L\phi \wedge L\psi \quad (7)$$
$$L(L\phi \vee L\psi) \equiv L\phi \vee L\psi \quad (8)$$
$$L(L\phi \supset L\psi) \equiv (L\phi \supset L\psi) \wedge (ML\phi \supset ML\psi) \quad (9)$$
$$ML\neg ML\phi \equiv \neg ML\phi \quad (10)$$
$$ML(\phi \wedge \psi) \equiv ML\phi \wedge ML\psi \quad (11)$$
$$ML(L\phi \vee L\psi) \equiv ML\phi \vee ML\psi \quad (12)$$
$$ML(L\phi \supset L\psi) \equiv ML\phi \supset ML\psi \quad (13)$$

3 NONMONOTONIC S4F

The most usual way of defining a nonmonotonic version of a modal logic is using McDermott and Doyle's fixpoint definition [14] of the concept of *expansion*. Given a modal logic $\mathcal{S}$, we say that theory E is an $\mathcal{S}$*-expansion* of theory T iff E is consistent with $\mathcal{S}$ and satisfies: $E = Cn_{\mathcal{S}}(T \cup \{\neg L\phi \mid \phi \notin E\})$.

In this work, however, we propose a different characterization in terms of minimal models. To this aim, we begin defining for any S4F model, an associated pair of sets formulas. We call *candidate set* to any consistent, logically closed set of classical formulas. For any S4F-model $\mathcal{M} = \langle W_1, W_2, V\rangle$ we define the pair of candidate sets $(H_{\mathcal{M}}, T_{\mathcal{M}})$ as:

$$H_{\mathcal{M}} \stackrel{\text{def}}{=} Th(I_w \mid w \in W_1 \cup W_2)$$
$$T_{\mathcal{M}} \stackrel{\text{def}}{=} Th(I_w \mid w \in W_2)$$

Looking at their definition, it is clear that $H_{\mathcal{M}} \subseteq T_{\mathcal{M}}$. A possible interpretation of $(H_{\mathcal{M}}, T_{\mathcal{M}})$ is that it describes a set of beliefs in a partial way: we *believe* all formulas in $H_{\mathcal{M}}$ whereas we *do not believe* any formula not in $T_{\mathcal{M}}$. Thus, there is no particular belief with respect to formulas in $T_{\mathcal{M}} - H_{\mathcal{M}}$.

This structure has a straightforward correspondence with modalities in S4F, as asserted by the following theorem:

Theorem 1 *For any classical formula F:* *($F \in H_{\mathcal{M}}$ iff $\mathcal{M} \models LF$) and ($F \in T_{\mathcal{M}}$ iff $\mathcal{M} \models MLF$).* □

Given two S4F models $\mathcal{M}, \mathcal{M}'$ we define the ordering relation: $\mathcal{M} \leq \mathcal{M}'$ iff $T_{\mathcal{M}} = T_{\mathcal{M}'}$ and $H_{\mathcal{M}} \subseteq H_{\mathcal{M}'}$. In other words, $\leq$-minimal models correspond to fixing the non-believed formulas and minimizing the believed ones.

Definition 1 (Selected model) An $\leq$-minimal S4F-model $\mathcal{M}$ of a theory T is said to be *selected* iff $H_{\mathcal{M}} = T_{\mathcal{M}}$. □

3.1 Intuitionistic Default Logic

We can now define an interesting subset of nonmonotonic S4F by restricting the use of modal operators in the following way. The language of *Intuitionistic Default Logic* (IDL) corresponds to $[not\ ,\ ;\ \leftarrow](\mathcal{L})$. In other words, we construct formulas with intuitionistic operators but using the set of classical formulas as "atoms". The name IDL should not lead to confusion: it is not an intuitionistic *variant* of Default Logic, but an intuitionistic *interpretation* of default constructs instead [4].

[4] The DL variant proposed in [15], for instance, not only interprets default rules as intuitionistic operators, but also proposes closing default theories under constructive rather than classical deduction.

We say that IDL is a subset of S4F because intuitionistic connectives are actually defined in terms of modal expressions, following Gödel's translation [6]:

$$(not\ \phi) \stackrel{\text{def}}{=} L\neg L\phi$$
$$(\phi, \psi) \stackrel{\text{def}}{=} \phi \wedge \psi$$
$$(\phi; \psi) \stackrel{\text{def}}{=} L\phi \vee L\psi$$
$$(\phi \leftarrow \psi) \stackrel{\text{def}}{=} L\psi \supset L\phi$$

The translation of negation[5] is equivalent to $\neg ML\phi$. Thus, by Theorem 1, an intuitive interpretation of $(not\ F)$ is $F \notin T_{\mathcal{M}}$, that is, F is *not believed* by the agent. It is not difficult to show that the following equivalences are theorems in S4F:

$$not\ not\ not\ \phi \equiv not\ \phi \quad (14)$$
$$not\ (\phi, \psi) \equiv (not\ \phi; not\ \psi) \quad (15)$$
$$not\ (\phi; \psi) \equiv (not\ \phi, not\ \psi) \quad (16)$$

The importance of these properties is that they show that formulas of sub-language $[not\ ,\ ;](\mathcal{L})$ can be unfolded until occurrences of *not* have the shape $(not\ F)$ or $(not\ not\ F)$, being F a classical formula.

It should perhaps be observed that, due to Proposition 1, requiring $\mathcal{M} \models \phi$ in IDL is the same than $\mathcal{M} \models L\phi$, and so, all expressions are implicitly in the scope of a necessity operator. The way in which this L operator can be unfolded with respect to intuitionistic operators is described by equivalences (1), (2) and (4)-(13).

The use of LP notation for intuitionistic operators allows establishing a direct syntactic correspondence with most classes of logic programs. For this reason, we have preferred to maintain the use of LP conjunction (ϕ, ψ), although as seen above, it does not differ from classical conjunction. In the case of variants of default theories the correspondence for our notation is not so straightforward, although it can be easily deduced. A disjunctive default rule like:

$$\frac{A : B_1, \ldots, B_n}{C_1 | \ldots | C_m}$$

would be represented in IDL as:

$$C_1; \ldots; C_m \leftarrow A, not\ \neg B_1, \ldots, not\ \neg B_n$$

The following property can be easily checked:

Property 1 *For any IDL formula ϕ, if $\langle W_1, W_2, V\rangle \models \phi$ then $\langle W_2, V\rangle \models \phi$.* □

That is, if a S4F model satisfies an IDL formula ϕ, then the S5-model just consisting of cluster W_2 also satisfies ϕ.

4 NESTED DEFAULT LOGIC

The syntax of Nested Default Logic (NDL) is a subset of IDL where connective '$\leftarrow$' cannot be in the scope of other operator. Therefore, a NDL theory is a set of *rules* like $\phi \leftarrow \psi$,

[5] For translating *not* ϕ, Gödel actually proposed a second alternative $\neg L\phi$. Although most results in the paper are still valid under this choice, we have preferred the stronger version $L\neg L\phi$ in order to obtain simpler proofs and provide a more direct interpretation of negation in terms of $T_{\mathcal{M}}$.

where ϕ and ψ belong to language $[not\ ,\ ;](\mathcal{L})$ (called the set of *NDL formulas*). Classical formulas can be included in the NDL theory as rules like $F \leftarrow \top$.

The *satisfaction* of an NDL formula ϕ by a candidate set X is denoted as $X \models_{\text{ND}} \phi$ and recursively defined as follows:

- $X \models_{\text{ND}} F$ iff $F \in X$, for any classical formula F.
- $X \models_{\text{ND}} (\phi, \psi)$ iff $X \models_{\text{ND}} \phi$ and $X \models_{\text{ND}} \psi$
- $X \models_{\text{ND}} (\phi; \psi)$ iff $X \models_{\text{ND}} \phi$ or $X \models_{\text{ND}} \psi$
- $X \models_{\text{ND}} not\ \phi$ iff $X \not\models_{\text{ND}} \phi$

As expected, X is a *model* of a NDL theory D, also written $X \models_{\text{ND}} D$, when $X \models_{\text{ND}} \psi$ implies $X \models_{\text{ND}} \phi$, for any rule $\phi \leftarrow \psi$ in D. The *reduct* of a NDL formula ϕ with respect to X, denoted as ϕ^X, is the result of replacing any maximal[6] subformula *not* ψ either by $\bot$ or $\top$ depending on whether $X \models_{\text{ND}} \psi$ or not, respectively. The *reduct* of a default theory D, written D^X, is obtained by replacing each rule $\phi \leftarrow \psi$ in D by $\phi^X \leftarrow \psi^X$.

Definition 2 (Extension) A candidate set X is an *extension* of a default theory D iff X is a minimal (w.r.t. set inclusion) model of D^X. □

In [21] it is shown that NDL properly generalizes Reiter's Default Logic and its extension for disjunctive defaults introduced in [5]. The following theorem shows that IDL semantics covers, in its turn, NDL extensions:

Theorem 2 *Let D be a NDL theory, X a candidate set and $\mathcal{M}$ a S4F model for which $H_{\mathcal{M}} = T_{\mathcal{M}} = X$. Then, X is an extension of D iff $\mathcal{M}$ is a selected model for D under IDL.* □

The structure of a single candidate set does not suffice, however, for capturing the property of strong equivalence of default theories.

Definition 3 (SE-model) We define a *SE-model* of some default theory D as a pair (X, Y) of candidate sets satisfying $X \subseteq Y$, $X \models_{\text{ND}} D^Y$ and $Y \models_{\text{ND}} D^Y$. □

The idea of handling these two sets is that we will take into account both the "initial" candidate set Y used for getting the reduct, and the "resulting" candidate sets X that are models of the reduct.

Proposition 2 (From Theorem 3 in [21]) *Two NDL theories are strongly equivalent iff they have the same SE-models.* □

Now, the following theorem is essential for adapting this result for IDL:

Theorem 3 *Let (X, Y) be a pair of candidate sets with $X \subseteq Y$ and let $\mathcal{M}$ be some S4F model such that $X = H_{\mathcal{M}}$ and $Y = T_{\mathcal{M}}$. Then, for any NDL theory D, (X, Y) is an SE-model of D iff $\mathcal{M} \models D$ in IDL.* □

This directly means that we can rephrase now Proposition 2 so that strong equivalence of NDL theories corresponds to S4F-equivalence of their modal translations. In fact, this result is even more general. Since Turner's proof for Proposition 2 exclusively deals with sets of SE-models (without reference to the syntax of their original theories), and thanks to correspondence established in Theorem 2, it is not difficult to see that:

[6] That is, any subformula (*not* ψ) of ϕ which is not, in its turn, in the scope of an outer *not* .

Corollary 1 *Two IDL theories are strongly equivalent iff their modal encodings are S4F equivalent.* □

5 HERE-AND-THERE

The previous section has shown that IDL generalizes NDL which, in its turn, is a generalization of nested LP. On the other hand, as said in the introduction, Lifschitz, Pearce and Valverde [10] showed that the HT encoding of LP also captures nested expressions. However, the HT encoding is still applicable to more general expressions than nested LP syntax, providing an intuitive meaning to constructions in which the rule arrow is inside the scope of other operator. For instance, in HT we have:

$$(p \leftarrow q) \leftarrow r \quad \Leftrightarrow \quad p \leftarrow q, r$$
$$not\ (p \leftarrow q) \quad \Leftrightarrow \quad (\bot \leftarrow not\ q),\ (\bot \leftarrow p)$$

where $\Leftrightarrow$ stands for semantic equivalence. This nice feature may be lost for other encodings, like shown for instance in [1], for the case of three-valued logic.

The question now is, when we move to consider classical formulas instead of atoms, does IDL provide an intuitive meaning for this type of constructions? In this section we show that, in fact, the monotonic basis of IDL (that is, its S4F translation) is a proper generalization of HT. In other words, Gödel's pattern for translating intuitionistic logic into S4 is also valid for translating HT into S4F.

We use the language $[not\ ,\ ;\ \leftarrow\ \bot](At)$, called *intuitionistic formulas*, for describing the syntax of Here-and-There (HT). We will understand *not* ϕ as an abbreviation of $\bot \leftarrow \phi$. The semantics of HT is described as follows. An HT *world* is any element of the set $\{h, t\}$ (respectively standing for *here* and *there*). We define an accessibility relation $\preceq$ so that $h \preceq h$, $t \preceq t$ and $h \preceq t$.

Definition 4 (HT-interpretation) Given a propositional signature At, an HT *interpretation* is defined as the pair $\mathcal{I} = (I^h, I^t)$ where $I^h \subseteq I^t \subseteq At$. □

The HT interpretation can be understood as a partial truth valuation for atoms in the signature. Intuitively, I^h contains the true atoms, $\Sigma - I^t$ the false atoms and, finally, $I^t - I^h$ corresponds to those atoms that are left undefined. An interpretation of shape (I, I) is said to be *total* (there are no undefined atoms).

Definition 5 (Satisfaction of a formula) We recursively define the *satisfaction* of a formula ϕ by an interpretation $\mathcal{I} = (I^h, I^t)$ at a world w, written $(\mathcal{I}, w) \models_{\text{HT}} \phi$, in the following way:

1. $(\mathcal{I}, w) \models_{\text{HT}} p$ iff $p \in I^w$
2. $(\mathcal{I}, w) \models_{\text{HT}} (\phi, \psi)$ iff $(\mathcal{I}, w) \models_{\text{HT}} \phi$ and $(\mathcal{I}, w) \models_{\text{HT}} \psi$
3. $(\mathcal{I}, w) \models_{\text{HT}} (\phi; \psi)$ iff $(\mathcal{I}, w) \models_{\text{HT}} \phi$ or $(\mathcal{I}, w) \models_{\text{HT}} \psi$
4. $(\mathcal{I}, w) \models_{\text{HT}} (\psi \leftarrow \phi)$ iff for all w' such that $w \preceq w'$, $(\mathcal{I}, w') \not\models_{\text{HT}} \phi$ or $(\mathcal{I}, w) \models_{\text{HT}} \psi$
5. $(\mathcal{I}, w) \not\models_{\text{HT}} \bot$

□

We say that an HT interpretation $\mathcal{I}$ is a *model* of a theory T iff $(\mathcal{I}, h) \models_{\text{HT}} \phi$ for all ϕ in T. The following property of HT corresponds, somehow, to Property 1 for IDL:

Property 2 *For all intuitionistic formula* ϕ, *if* $(\mathcal{I}, h) \models \phi$ *then* $(\mathcal{I}, t) \models \phi$. □

Definition 6 *Given a S4F model* $\mathcal{M}$ *we define the corresponding HT interpretation* $\mathcal{I}_{\mathcal{M}} = (I^h_{\mathcal{M}}, I^t_{\mathcal{M}})$ *as:*

$$I^h_{\mathcal{M}} \stackrel{\text{def}}{=} \bigcap_{w \in W_1 \cup W_2} I_w \qquad\qquad I^t_{\mathcal{M}} \stackrel{\text{def}}{=} \bigcap_{w \in W_2} I_w$$

□

In other words, $I^h_{\mathcal{M}}$ (resp. $I^t_{\mathcal{M}}$) collects all the atoms included in $H_{\mathcal{M}}$ (resp. $T_{\mathcal{M}}$). Note that different S4F models may lead to the same $I_{\mathcal{M}}$.

Lemma 1 *Let* ϕ *be an intuitionistic formula,* $\mathcal{M} = \langle W_1, W_2, V\rangle$ *an S4F model and* $I_{\mathcal{M}}$ *its corresponding HT interpretation. Then:*
(a) $(\mathcal{I}_{\mathcal{M}}, h) \models_{HT} \phi$ *iff* $\mathcal{M} \models \phi$.
(b) $(\mathcal{I}_{\mathcal{M}}, t) \models_{HT} \phi$ *iff* $\langle W_2, V\rangle \models \phi$. □

Theorem 4 *For any intuitionistic formula* ϕ:
$\models_{HT} \phi$ *iff* $\models \phi$ *under S4F.* □

6 CONCLUSION

We have reconsidered the S4F encoding of Default Logic (DL) by first interpreting default constructs as intuitionistic operators (we called this *Intuitionistic* DL) and then using Gödel's translation into modal logic. This orientation has allowed us, for instance, to capture Turner's Nested DL (NDL) and generalize his result for characterizing strong equivalence of default theories. Besides, some important new advantages can be obtained with respect to NDL. As an example, we can show properties about default theories in terms of theorem proving in S4F, what can be automated with a tableaux-style prover (like the one proposed in [7], for instance). Note that, on the contrary, when we directly use NDL definitions, automated reasoning is not straightforward: (meta)proofs for properties are *ad hoc*, using non-logical constructs (like the theory reduct) and logically closed sets of formulas. Another advantage is that our approach still provides a meaning for expressions where the rule conditional is in the scope of other operators. Thus, we can really *nest default rules*, something not possible in NDL, and preserve the same meaning than the one provided by Here-and-There with respect to logic programs. In fact, the relation we established between S4F and Here-and-There, allows using modal S4F provers for proving theorems in that intermediate logic.

Some open topics are left for future work. For instance, it remains to prove that the S4F models minimization proposed in this paper actually corresponds to the standard McDermott and Doyle's syntactic fixpoint definition. Another interesting topic is the relation to the bimodal logic of *Minimal Belief and Negation as Failure* (MBNF) [9]. Our conjecture is that MBNF is weaker than S4F, and this weakness will prevent to obtain a necessary condition for strong equivalence of default theories.

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