Lower Bound Founded Logic of Here-and-There

Sebastian Schellhorn

University of Potsdam
Institute of Computer Science
Knowledge Representation and Reasoning

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Outline

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Motivation

- in *ASP* atoms of stable models must be founded
- stable models can be defined by the logic of Here-and-There (*HT*) and equilibrium models
- *foundedness*: regard $\mathbf{t} \geq \mathbf{f}$ and assign smallest truth value that can be proven

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- in *ASP* atoms of stable models must be founded
- stable models can be defined by the logic of Here-and-There (HT) and equilibrium models
- *foundedness*: regard $\mathbf{t} \geq \mathbf{f}$ and assign smallest truth value that can be proven

Example

$$a$$
$$b \leftarrow c$$

We have stable model $\{a\}$.

Motivation

This idea of foundedness was generalized to ordered domains.

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Examples

$$x \geq 1$$
$$x \geq 42 \leftarrow \neg(x \leq 1)$$

Regarding foundedness we expect solutions $x \mapsto 1$ and $x \mapsto 42$.

$$x \geq 0$$
$$y \geq 0$$
$$x \geq 42 \leftarrow y \leq 42$$

Regarding foundedness we expect the solution with $x \mapsto 42$ and $y \mapsto 0$.

Motivation

This idea of foundedness was generalized to ordered domains.

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Regarding foundedness we expect the solution with $x \mapsto 42$ and $y \mapsto 0$.

Outline

Logic of Here-and-There (HT) (1)

- set of propositional atoms $\mathcal{A}$
- formula: combination of propositional atoms and logical connectives $\bot$, $\wedge$, $\vee$, $\leftarrow$
- theory: a set of formulas
- interpretation: a set of atoms
- HT-interpretation: a pair $\langle H, T\rangle$ of interpretations with $H \subseteq T$
- denotation: $[\![\,\cdot\,]\!]_{\mathcal{A}} : \mathcal{A} \rightarrow 2^{\mathcal{A}}$, that is $[\![\,p\,]\!]_{\mathcal{A}} \stackrel{\text{def}}{=} \{I \mid p \in I\}$ for $p \in \mathcal{A}$

Logic of Here-and-There (HT) (2)

- satisfaction of formula φ over $\mathcal{A}$ by HT-interpretation $\langle H, T\rangle$:
  1. $\langle H, T\rangle \not\models \bot$
  2. $\langle H, T\rangle \models p$ iff $H \in [\![p]\!]_{\mathcal{A}}$ for propositional atom $p \in \mathcal{A}$
  3. $\langle H, T\rangle \models \varphi_1 \wedge \varphi_2$ iff $\langle H, T\rangle \models \varphi_1$ and $\langle H, T\rangle \models \varphi_2$
  4. $\langle H, T\rangle \models \varphi_1 \vee \varphi_2$ iff $\langle H, T\rangle \models \varphi_1$ or $\langle H, T\rangle \models \varphi_2$
  5. $\langle H, T\rangle \models \varphi_1 \rightarrow \varphi_2$ iff $\langle I, T\rangle \not\models \varphi_1$ or $\langle I, T\rangle \models \varphi_2$ for both $I \in \{H, T\}$
- $\langle T, T\rangle$ equilibrium model of theory Γ over $\mathcal{A}$iff $\langle T, T\rangle \models \Gamma$ and there is no $H \subset T$ with $\langle H, T\rangle \models \Gamma$

HT_{LB} (1)

- set of atoms $\mathcal{A}_{\mathcal{X}}$, comprising variables $\mathcal{X}$ and constants over ordered domain $(\mathcal{D}, \succeq)$
- formula: combination of atoms and logical connectives $\bot$, $\wedge$, $\vee$, $\leftarrow$
- value $\boldsymbol{u}$ stands for *undefined*
- valuation: $v : \mathcal{X} \rightarrow \mathcal{D}_{\boldsymbol{u}}$
- $\mathfrak{V}_{\mathcal{X},\mathcal{D}}$ represents the set of valuations
- HT_{LB}-valuation over $\mathcal{X}$, $\mathcal{D}$: a pair $\langle h, t \rangle$ of valuations with $h \subseteq t$
- denotation: $[\![\cdot]\!]_{\mathcal{X},\mathcal{D}} : \mathcal{A}_{\mathcal{X}} \rightarrow 2^{\mathfrak{V}_{\mathcal{X},\mathcal{D}}}$, eg $[\![x \geq 42]\!] \stackrel{\text{def}}{=} \{v \mid v(x) \geq 42\}$

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Example
Consider variables x and y with domain $\{0, 1, 2, 3\} \cup \{\boldsymbol{u}\}$

$$v = \{x \mapsto 2, y \mapsto 0\} \text{ and } v' = \{x \mapsto 1\}$$

can be represented by

$$v = \{(x, 0), (x, 1), (x, 2), (y, 0)\} = (x{\downarrow}2) \cup (y{\downarrow}0) \text{ and } v' = (x{\downarrow}1)$$

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HT_{LB} (2)

- satisfaction of formula φ over $\mathcal{A}_{\mathcal{X}}$ by HT_{LB}-valuation $\langle h, t \rangle$:
  1. $\langle h, t \rangle \not\models \bot$
  2. $\langle h, t \rangle \models a$ iff $v \in [\![a]\!]_{\mathcal{X},\mathcal{D}}$ for atom $a \in \mathcal{A}_{\mathcal{X}}$ and for both $v \in \{h, t\}$
  3. $\langle h, t \rangle \models \varphi_1 \wedge \varphi_2$ iff $\langle h, t \rangle \models \varphi_1$ and $\langle h, t \rangle \models \varphi_2$
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- $\langle t, t \rangle$ equilibrium model of theory Γ over $\mathcal{A}_{\mathcal{X}}$ iff $\langle t, t \rangle \models \Gamma$ and there is no $h \subset t$ with $\langle h, t \rangle \models \Gamma$

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HT_{LB} Results

Proposition (Persistence and Negation)

Let $\langle h, t\rangle$ and $\langle t, t\rangle$ be HT_{LB}-valuations over $\mathcal{X}, \mathcal{D}$, and φ be a formula over $\mathcal{A}_{\mathcal{X}}$. Then,

1. $\langle h, t\rangle \models \varphi$ *implies* $\langle t, t\rangle \models \varphi$, *and*
2. $\langle h, t\rangle \models \varphi \rightarrow \bot$ *iff* $\langle t, t\rangle \not\models \varphi$.

HT_{LB} Results

Proposition (Strong Equivalence)

Let Γ_1, Γ_2 and Γ be theories over $\mathcal{A}_{\mathcal{X}}$. Then, theories $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ have the same HT_{LB}-stable models for every theory Γ iff Γ_1 and Γ_2 have the same HT_{LB}-models.

HT_{LB} Results

Relating HT and HT_{LB}:

- HT can be seen as a special case of HT_{LB}.
- every model in HT_{LB} induces a model in HT.
- every tautology in HT is a tautology in HT_{LB}.

HT_{LB} Results

By straightforward defining classical satisfaction and the reduct of Ferraris in our setting we get:

Proposition (Stable models and equilibrium models coincide)

Let $\langle h, t \rangle$ be an HT_{LB}-valuation over $\mathcal{X}, \mathcal{D}$ and Γ a theory over $\mathcal{A}_{\mathcal{X}}$. Then, $h \models_{cl} \Gamma^t$ iff $\langle h, t \rangle \models \Gamma$.

Outline

Linear Constraint Atoms

- linear constraint atom: $\sum_{i=1}^{m} w_i x_i \prec k$
  where $w_i, k \in \mathbb{Z}$ constants, x_i variables, and $\prec \in \{\geq, \leq, \neq, =\}$ a binary relation
- $\mathcal{L}_{\mathcal{X}}$ set of linear constraint atoms
- denotation: $[\![\sum_{i=1}^{m} w_i x_i \prec k]\!] \stackrel{\text{def}}{=} \{v \mid \sum_{i=1}^{m} w_i v(x_i) \prec k,\ v(x_i) \neq \boldsymbol{u}\}$

Programs

- rule: $a_0 \vee \cdots \vee a_n \leftarrow a_{n+1} \wedge \cdots \wedge a_{n'} \wedge \neg a_{n'+1} \wedge \cdots \wedge \neg a_{n''}$ with a_i atoms of $\mathcal{L}_{\mathcal{X}}$ for $0 \leq i \leq n''$
- program: a set of rules

Example

$$x + y \geq 42$$
$$x \geq 0$$
$$x \geq 42 \leftarrow \neg(x \leq 1)$$

Monotonicity

- We define an atom a as *monotonic* (resp. *anti-monotonic*) wrt variable x if $v \in [\![a]\!]$ implies $v' \in [\![a]\!]$ for every valuation v' with $v \subseteq v'$ (resp. $v' \subseteq v$ with $v'(x) \neq \boldsymbol{u}$), where $v(y) = v'(y)$ for all $y \in vars(a) \setminus \{x\}$.
- We define an atom a as *monotonic* (resp. *anti-monotonic*) if it is monotonic (resp. anti-monotonic) wrt all variables in $vars(a)$, and non-monotonic otherwise.

Normal Programs

- normal rule: $a_0 \leftarrow a_1 \wedge \cdots \wedge a_n \wedge \neg a_{n+1} \wedge \cdots \wedge \neg a_{n'}$
  with a_i atoms of $\mathcal{L}_{\mathcal{X}}$ for $0 \leq i \leq n'$ and $|vars(a_0)| = 1$ a_j monotonic for $n+1 \leq j \leq n'$
- normal program: a set of normal rules

Proposition

Let P be a normal program over $\mathcal{L}_{\mathcal{X}}$. Then, each HT_{LB}-stable model of P over $\mathcal{X}, \mathbb{Z}$ is subset minimal.

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Example of not normal program

$$x + y \geq 42$$

infinite many stable models $\{v \mid v(x) + v(y) = 42\}$

Normal Programs

- normal rule: $a_0 \leftarrow a_1 \wedge \cdots \wedge a_n \wedge \neg a_{n+1} \wedge \cdots \wedge \neg a_{n'}$
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Let P be a normal program over $\mathcal{L}_{\mathcal{X}}$. Then, each HT_{LB}-stable model of P over $\mathcal{X}, \mathbb{Z}$ is subset minimal.

Example of not normal program

$$x \geq 0$$
$$x \geq 42 \leftarrow \neg(x \leq 1)$$

stable models $(x{\downarrow}1)$ and $(x{\downarrow}42)$, where $(x{\downarrow}1) \subset (x{\downarrow}42)$

Positive Programs

- positive body: r normal rule, then $body^{+}(r) \stackrel{\text{def}}{=} \{a_i \mid 1 \leq i \leq n, a_i \text{ monotonic}\}$
- negative body: r normal rule, then $body^{-}(r) \stackrel{\text{def}}{=} body(r) \setminus body^{+}(r)$
- positive rule: is a normal rule with $head(r)$ monotonic and $body^{-}(r) = \emptyset$
- positive program: a set of positive rules

Proposition

Let P be a positive program over $\mathcal{L}_{\mathcal{X}}$. Then, P has exactly one HT_{LB}-stable model over $\mathcal{X}, \mathbb{Z}$.

Positive Programs

- positive body: r normal rule, then $body^+(r) \stackrel{\text{def}}{=} \{a_i \mid 1 \leq i \leq n, a_i \text{ monotonic}\}$
- negative body: r normal rule, then $body^-(r) \stackrel{\text{def}}{=} body(r) \setminus body^+(r)$
- positive rule: is a normal rule with $head(r)$ monotonic and $body^-(r) = \emptyset$
- positive program: a set of positive rules

Proposition

Let P be a positive program over $\mathcal{L}_{\mathcal{X}}$. Then, P has exactly one HT_{LB}-stable model over $\mathcal{X}, \mathbb{Z}$.

Example of not positive program

$$x \geq 0 \qquad x \geq 42 \leftarrow y < 42$$
$$y \geq 0 \qquad y \geq 42 \leftarrow x < 42$$

with stable models $(x{\downarrow}42) \cup (y{\downarrow}0)$ and $(x{\downarrow}0) \cup (y{\downarrow}42)$

Stratified Programs

- idea: free of recursion through negation

Proposition

Let P be a stratified program over $\mathcal{L}_{\mathcal{X}}$ with monotonic heads only. Then, P has exactly one HT_{LB}-stable model over $\mathcal{X}, \mathbb{Z}$.

Proposition

Let P be a stratified program over $\mathcal{L}_{\mathcal{X}}$. Then, P has at most one HT_{LB}-stable model over $\mathcal{X}, \mathbb{Z}$.

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Let P be a stratified program over $\mathcal{L}_{\mathcal{X}}$. Then, P has at most one HT_{LB}-stable model over $\mathcal{X}, \mathbb{Z}$.

Example

$$x \geq 42$$
$$x < 42$$

no stable model

Outline

Bound Founded *ASP* (*BFASP*)

BFASP is different to HT_{LB}, since it

- not distinguishes monotonicity of atoms and logic connectives
- sets valuations per default to the smallest domain value (may different from undefined)
- understands $x + y \geq 42$ as an implication
- has unintuitive stable model $\{p\}$ for $p \leftarrow \neg p$

Logic of Here-and-There with Constraints (HT_C)

HT_C is different to HT_{LB}, since it

- not compares valuations wrt the values assigned to the variables
- not minimizes valuations wrt foundedness
- allows atoms with closed denotations only

Both HT_C and HT_{LB}

- base on HT
- capture theories over constraint atoms in a non-monotonic setting
- easily express default values

Other

Integer Linear Programming (*ILP*)

- monotone theory
- not intuitive to model recursion

Other

ASP modulo Theory

- integrate monotone theories in the non-monotonic setting of *ASP*
- stable models rely on any possible valid assignment for variables

Other

Aggregates

- extensions of *ASP* allowing to perform set operations on elements of a respective set
- aggregates under Ferraris' semantics can be represented as atoms in HT_{LB}

Outline

Conclusion

HT_{LB}

- provides foundedness for minimal values of variables over ordered domains
- preserves persistence, negation and strong equivalence
- generalizes HT
- agrees to a Ferraris-like stable models semantics
- generalizes concepts like normal, stratified and positive programs and preserves corresponding properties
- improves or generalizes existing approaches