Logic.
Computational Complexity

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1 Computational Complexity

Turing Machine

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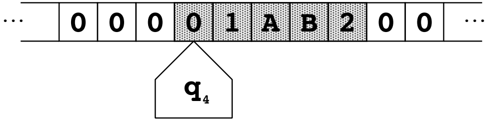
(1912-1952)



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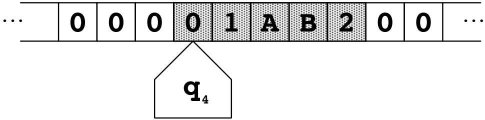

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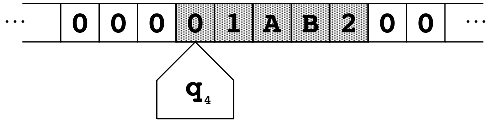

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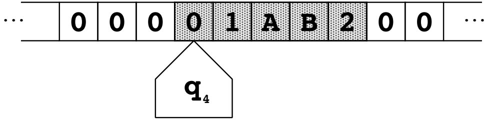
- Its transition function describes jumps from state to next state.

Transition function

- Example: with scanned symbol 0 and state q_4, write 1, move *Left* and go to state q_2. That is:



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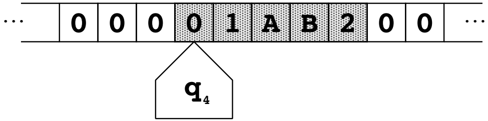

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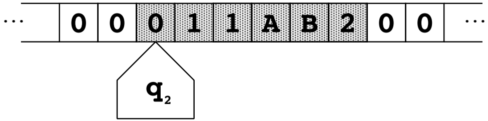

Decision problems

Definition (Decision problem)

A decision problem consists in providing a given tape input and asking the TM for a final output symbol answering *Yes* or *No*.

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- Example: $HALTING$ = given another TM M plus its input tape T, answer *yes* if M, T stops
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- Example: $\overline{SAT} = UNSAT$ answers *no* if the formula has a model.

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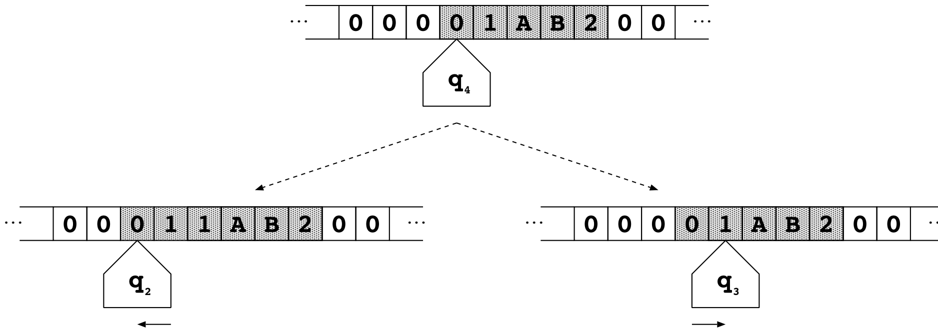
- Example: $t(0, q_4, 1, Left, q_2)$, $t(0, q_4, 0, Right, q_3)$



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  The sequence of these two steps takes polynomial time.

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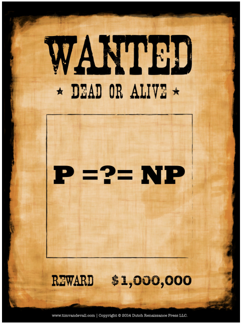
It is one of the 7 Millenium Prize Problems

http://www.claymath.org/millennium-problems



The Clay Mathematics Institute designated $1 million prize for its solution!

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- The Complexity Zoo
  https://complexityzoo.uwaterloo.ca/Complexity_Zoo

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- In general, **co−NP** $\neq$ **NP** (the intersection is non-empty)
- *UNSAT* is in **co−NP**. This implies that *VAL* (deciding whether α is valid) is also **co−NP**.

Exercise: Turing machine in Prolog

- We use `tape(Ls,S,Rs)` to represent the current symbol `S`, the left fragment of the tape `Ls` (reversed) and the right one `Rs`.

```
compute(Q, T, T) :- final(Q), !.

compute(Q0, tape(Ls0,S,Rs0), T):-
  showmachine(Q0,Ls0,S,Rs0),
  t(Q0,S,  Q1,S1,Action),
  move(Action,tape(Ls0,S1,Rs0),T1),
  compute(Q1,T1,T).

move(l,tape([],    S,Rs), tape([],0,[S|Rs])).
move(l,tape([L|Ls],S,Rs), tape(Ls,L,[S|Rs])).

move(r,tape(Ls,S,[]),     tape([S|Ls],0,[])).
move(r,tape(Ls,S,[R|Rs]), tape([S|Ls],R,Rs)).
```